Physics 718 Problem Set 2

Due: Feb 25, 2019

Problem 1 Euler and divergence of the Newtonian stress tensor

The Newtonian stress tensor of a fluid is $T^{ab}_{\text{fluid}} = \rho v^a v^b + P g^{ab}$, where g^{ab} is the flat Euclidean metric, with components δ^{ij} in Cartesian coordinates.

a. Show that Euler's equation is equivalent to

$$\partial_t(\rho v^a) + \nabla_b T^{ab}_{\text{fluid}} = -\rho \nabla^a \Phi, \tag{1}$$

assuming conservation of mass.

b. Now add the stress tensor of the gravitational field,

$$T_{G\,ab} = \frac{1}{4\pi G}\left(\nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} \nabla_c \Phi \nabla^c \Phi\right), \tag{2}$$

where $\nabla^2 \Phi = 4\pi G \rho$. Show that

$$\partial_t(\rho v^a) + \nabla_b T^{ab} = 0, \tag{3}$$

where $T^{ab} = T^{ab}_{\text{fluid}} + T^{ab}_G$.

Problem 2 Relativistic uniform-density stars

Solve the equilibrium equations,

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}$$
$$\frac{d\Phi}{dr} = -\frac{1}{(\rho + P)}\frac{dP}{dr} = \frac{(m + 4\pi r^3 P)}{r(r - 2m)}$$
$$\frac{dm}{dr} = 4\pi r^2 \rho,$$

for a spherical star of uniform density ρ. Note that Φ is continuous across the boundary of the star and vanishes as $r \to \infty$. You can check your solution against that given in Shapiro-Teukolsky (and in MTW).

Problem 3 Gravitational units to conventional units

In gravitational units, the dimension of every quantity is a power of L. Find the conversion factors (products of powers of G and c that convert the following quantities in gravitational units back to conventional units:
mass M, gravitational potential Φ, velocity v, pressure P, density ρ, energy density ε.
In each case, exhibit the gravitational dimension of each quantity as a power of L and show that the conversion yields the appropriate dimension as a product of powers of M, L, T.

Problem 4 Newtonian approximation

(MTW Sect. 17.4 is a detailed discussion of the Newtonian limit in gravitational units with $G = c = 1$.) The Newtonian approximation is the leading term in an expansion in the small dimensionless parameter v/c where $v^2 \sim GM/R$ with R a radius of a star (or a characteristic length for which $\nabla P \sim P/R$). The pressure in a Newtonian star and the Newtonian internal energy density are each of order ρv^2. Particles in the star move with speeds of order v and objects orbiting the star move with speeds no larger than this. Then

$$\begin{aligned} \epsilon &= \rho c^2 = \rho_0 c^2 + \varepsilon, \quad \varepsilon = O(\rho_0 v^2), \\ \frac{GM}{r} &= O(v^2), \quad P = O(\rho v^2), \\ \Phi &= O(v^2). \end{aligned} \tag{4}$$

Show that the Newtonian equations of hydrostatic equilibrium for a spherical star are the Newtonian limit of the relativistic equations (III.84)-(III.86) of the notes (or (5.7.5)-(5.7.7) of Shapiro-Teukolsky). The terms you discard are smaller than the terms you keep by some power of v/c. For each term you throw away, give the ratio of the term you keep to the term you discard as a power of v/c.

Problem 5 Lane Emden and Polytropes

If you did this problem already in Astron 400, try to solve the relativistic case instead. Compare your solution for $n = 0$ against Problem 2.

Use a numerical method to compute the density profiles $\rho(r)$ for polytropes with $n = 0, 1, 1.5, 3, 5$. Be sure to correctly incorporate the boundary conditions in your integrations. Plot these profiles, and compare them with the anlytic solutions for $n = 0, 1, 5$. Be sure you only plot out to the first zero.

Please email your code and plots to `kaplan@uwm.edu`.

Problem 6 Empirical Properties of Stars

Using the data below, study various empirical properties of stars.

Please email your code and plots to `kaplan@uwm.edu`.

a. Find the variation of M/R along the main sequence. Assuming that $M/R \propto M^\sigma$ is an adequate fit, estimate σ. [Hint: to fit a relation of this type, you want to fit $y = ax^b$ and solve for a and b. This is a power-law fit. The easiest way to do this is to take the log of both sides, and then it becomes a linear fit.]

b. The mean density is $\bar{\rho}(M) = 3M/4\pi R^3$. The central density $\rho_c(M)$ is roughly proportional to $\bar{\rho}(M)$. Determine the variation of $\rho_c(M)$ with M along the main sequence based on your results from part (a) [Hint: you should not have to fit any more data]. You can normalize this relation by using the fact that for the Sun, $\rho_c \approx 1.5 \times 10^5\,\mathrm{kg\,m^{-3}}$.

c. Let ε be the energy generated per kg per second in a main sequence star. Assume that $\varepsilon \propto \rho_c(M)(M/R)^\alpha$. Estimate α in a simiilar manner to what you did in part (a).

d. We know that H-burning is responsible for the energy generation in a main sequence star. This involves conversion of four protons in the center of the star into one ^{4}He nucleus, releasing about 27 MeV per ^{4}He nucleus produced. Assuming that a fraction of $f \approx 0.1$ of the entire mass of the star is converted from H into He on the main sequence, estimate the main sequence lifetime as a function of stellar mass using your results from part (c). [Hint: you should not have to fit any more data].

Properties of Main Sequence Stars

$M\ (M_\odot)$	$\log_{10}(L/L_\odot)$	T (K)
23	5.2	37,000
16	4.6	30,500
10	4.0	24.000
7	3.6	17,700
4.5	2.7	14,000
3.6	2.2	11,800
3.1	1.9	10,500
2.7	1.6	9,500
2.3	1.3	8,500
1.9	1.0	7,900
1.6	0.8	7,350
1.4	0.6	6,800
1.25	0.3	6,300
1.03	0.1	5,900
0.91	−0.1	5,540
0.83	−0.3	5,330
0.77	−0.5	5,090
0.72	−0.6	4,840
0.67	−0.7	4,590
0.62	−0.8	4,350
0.56	−0.9	4,100
0.50	−1.1	3,850