# Astron 400 Problem Set 4

Given: Sep 29. Due: Thursday, Oct 13 at the beginning of class

**Homework Policy:** You can consult class notes and books. Always try to solve the problems yourself; if you cannot make progress after some effort, you can discuss with your classmates or ask the instructor. However, you cannot copy other's work: what you turn in must be your own. Make sure you are clear about the process you use to solve the problems: partial credit will be awarded.

**Reading:** Phillips Chapter 3, 4

## Problem 1 Mass-Luminosity Relations: Theoretical

The gas pressure $P_g$ dominates over radiation pressure $P_r$ for stars with $M < 30\,M_\odot$. Free-free Kramer's opacity $\kappa_{ff}$ dominates over electron scattering $\kappa_{es}$ when $M < 5\,M_\odot$.

Start with the ideal gas law and the definition of radiation pressure. Use hydrostatic equilibrium and the equation for radiative energy transport. Show that luminosity $L$ and mass $M$ have the following approximate scaling relations:

a. For low-mass stars with $\ll 5\,M_\odot$ (so free-free opacity and gas pressure dominate), show $L \propto M^5$ (this ignores convection, which can be important in low-mass stars).

b. For intermediate masses $5-30\,M_\odot$ (so gas pressure and electron scattering dominate), show $L \propto M^3$.

c. For high masses $\gg 30\,M_\odot$ (radiation pressure and electron scattering), show $L \propto M$

## Problem 2 Neutron Star Crust

The outer layer of a neutron star consists of $^{56}$Fe ions embedded in a sea of electrons. [It is convenient to remember that $1\,\mathrm{eV} = 1.16 \times 10^4\,\mathrm{K}$, $\hbar c = 197.3\,\mathrm{MeV\,fm}$.]

a. What is the electron fraction $Y_e \equiv n_e/n_b$, where $n_b$ is the baryon number density $n_b = \rho/m_p$? [Baryons are protons and neutrons, and you can ignore the difference between proton and neutron masses].

b. Show that the electron Fermi momentum $p_F$ is given by:

$$p_F c = 1.11(Y_e \rho_{10})^{1/3}\,\mathrm{MeV}$$

where $\rho_{10} = \rho/(10^{10}\,\mathrm{kg\,m^{-3}})$. At what density $\rho = \rho_r$ do the electrons start to become relativistic?

c. Sometimes it is convenient to define a "Fermi Temperature", $T_F = E_F/k_B$. Physically, $T_F$ is the temperature below which electrons are degenerate. Plot $T_F$ as a function of $\rho$ for the density range $10^3 - 10^{10}\,\mathrm{kg\,m^{-3}}$.

d. What is the Fermi energy for the ions as a function of density, over the same range as part (c)? Think about this a bit: what type of particle are the ions, bosons or fermions? What statistics do they obey?

## Problem 3 Electron-Positron Creation

At high temperatures, photons can convert to electron-positron pairs and an equilibrium is established:

$$\gamma + \gamma \leftrightarrow e^+ + e^-$$

Recall that photons always have 0 chemical potential.

a. For $T \ll m_e c^2/k_B$, we can treat the electrons and positrons as non-relativistic particles. Since the pair density is rather low, the pairs are very non-degenerate. Show that in equilibrium:

$$n_- n_+ = 4\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^3 e^{-2m_e c^2/k_B T}$$

where $n_+$ and $n_-$ are the number densities of positrons and electrons, respectively.

b. In general, $n_- \neq n_+$ because the medium may also contain ions, and charge neutrality requires $n_- = n_+ + Zn_i$ ($Z$ is the ion charge and $n_i$ the ion number density). As $T$ increases, the density of electrons from pair production becomes much larger than the density of electrons associated with ions, and then we have $n_- = n_+$ to a good approximation. Argue that in this case $\mu(e^+) = \mu(e^-) = 0$.

c. When $T \gg m_e c^2/k_B \approx 6\times 10^9$ K the electrons and positrons are extremely relativistic. Show that

$$n_+ = n_- = \frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty dx \frac{x^2}{e^x + 1}$$

To do this, go back to Eqn. 2.18 for the number of particles in a gas with energy $\epsilon_p$ and temperature $T$. Use this to get an expression for the number density, and then assume $k_B T \gg m_e c^2$ and that electrons/positrons are relativistic so $\epsilon_p = pc$.

Compare this to the number density of photons, $n_\gamma$. Which is larger, $n_\gamma$ or $n_-$? [**HINT:** you do not need to evaluate the integral to know the answer.]

## Problem 4 Phillips 3.3

## Problem 5 Phillips 4.1

## Problem 6 GS: Monte-Carlo Integration

A powerful technique for integrating problematic functions is known as Monte-Carlo integration. First developed during the Manhattan project, it uses random numbers to integrate functions. The simplest way is:

- Given an integral:
$$I = \int_{x_{\rm min}}^{x_{\rm max}} dx f(x)$$
- We sample $N$ uniform random numbers over the interval $[x_{\rm min}, x_{\rm max}]$, $x_i$
- We then approximate the integral by:
$$I \approx \frac{x_{\rm max} - x_{\rm min}}{N} \sum_{i=1}^{N} f(x_i)$$

There are fancier ways to do this, as you can see at:
`http://en.wikipedia.org/wiki/Monte_Carlo_integration`.

We will do Monte-Carlo integration on the integral from Problem 3c:

$$I = \int_0^\infty dx \, \frac{x^2}{e^x + 1}$$

Now, we know analytically that this is $3\zeta(3)/2 \approx 1.803085354$, where $\zeta(x)$ is the Riemann zeta function. But let's try to do this numerically.

One problem here is that the limits of integration are infinite. But we can't pick $x_{\rm max} = \infty$. So we need to try something else. If you plot the integrand $f(x) = x^2/(e^x + 1)$ you will see that it peaks near $x = 2$ and decreases steadily after that. So some value of $x_{\rm max}$ that is $> 2$ will work. Another question is what value of $N$ is best.

a. Write a routine that will do a Monte-Carlo integration of the integrand above for a given value of $N$ and $x_{\rm max}$.

b. You should run this $M = 100$ times and determine how well your integration does. i.e., compare your results to the exact result given above. The most useful quantity to compute is:
$$e(N, x_{\rm max}) = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \left[ \frac{I_j - \frac{3}{2}\zeta(3)}{\frac{3}{2}\zeta(3)} \right]^2}$$
This is the root-mean-square (rms) fractional error.

c. Repeat this for a range of $x_{\max}$ and $N$. How does $e(N, x_{\max})$ depend on $N$? How does it depend on $x_{\max}$ (plot them!). Does that make sense? What happens to your results as you increase $x_{\max}$ and why? This problem led to solutions such as the Metropolis-Hastings algorithm.