Astronomy 400: Astrophysics I

Physics 903: Stellar Astrophysics

Lecture I Preliminaries

Course Description: *quantitative* astronomy. Emphasis on the "why" and the "how" rather than just the "what." Will explore the structure and evolution of stars.

- Hydrostatic equilibrium, pressure support, gravitational collapse
- Virial theorem
- Nuclear fusion, energy generation in the Sun
- Radiation, interaction of radiation and matter
- Stellar evolution, on and off the main sequence
- Stellar remnants
- Explosions
- Star formation
- The following semester (Astron 401) will extend this discussion beyond the Milky Way.

The textbook will be Phillips: *The Physics of Stars* (2nd edition)

Evaluation will be:

- Weekly problem sets (50%), with the best 10 of 11 counting.
- Midterm exam (20%)
- Final exam (30%)

For graduate students, each problem set will have a numerical problem. For undergraduates that problem will be extra-credit.

I.2 Prerequisites

- Physics 309(P) (Physics 317 is preferred)
- Astron 103, 211 or permission of instructor.

I.2.1 Greek

If I use a symbol you don't recognize or can't read, **ask**!

I.3 Precision

We often do not know things very precisely. So we use $\sim$ and $\approx$ and related symbols. $\sim$ is for when we know something to *an order of magnitude*. So we if we know that $x \sim 5$, we know that x is between $5/3$ and $5 * 3$, where 3 is roughly $\sqrt{10}$. This means that the possible range for x is in total a factor of 10. We will also sometimes use $\sim$ to mean *scales as*. For example, if you were to estimate the height of a person as a function of their weight (for a wide range of people), you might expect that as you double the weight, the height changes by $2^{1/3}$. We could write height$\sim$weight$^{1/3}$. There will be a lot of variation, but this is roughly correct.

$\approx$ means more precision. It doesn't necessarily have an exact definition. But generally, if we say $x \approx 5$, that means that 4 is probably OK but 2 is probably not.

Finally, we have $\propto$, which means *proportional to*. This is more precise that the *scales as* use of $\sim$. So while for a person height$\sim$weight$^{1/3}$ is OK, for a sphere (where we know that volume is $4\pi/3r^3$) we could write volume$\propto r^3$: we take this as correct, but leave off the constants ($4\pi/3$ in this case).

I.3.1 Small Angles

For small angles θ, $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. We need θ to be in radians. But we also often deal with fractions of a circle. A circle has $360°$. We break each degree into 60 minute (or *arcminutes*): $1° = 60'$. And each arcminute into 60 seconds (or *arcseconds*): $1' = 60''$, so $1° = 3600''$. But we also know that 2π radians is $360°$, so we can convert between radians and arcsec. This will come up frequently: $1'' = 360 \times 3600/2\pi \approx 1/206265$ radians.

I.4 Celestial Sizes, Distances, and Coordinates

I.4.1 Units

Astronomy emphasizes *natural* units ($\odot$ is for the Sun, $\oplus$ is for the Earth):

- $M_\odot = 2 \times 10^{30}\,$kg (solar mass)
- $R_\odot = 7 \times 10^{8}\,$m (solar radius)
- $M_\oplus = 6 \times 10^{24}\,$kg $\approx 3 \times 10^{-6}\,M_\odot$ (earth mass)
- $M_{\rm J} = 2 \times 10^{27}\,$kg $\approx 10^{-3}\,M_\odot$ (Jupiter)
- $L_\odot = 4 \times 10^{26}\,$W (solar luminosity or power)
- light year $= 10^{16}$ m: the *distance* light travels in one year (moving at $c = 3 \times 10^8\,{\rm m\,s^{-1}}$)
- Astronomical Unit = AU = 1.5×10^{11} m (distance between earth and sun)

- parsec = parallax second (we will understand this later) = pc = $3 \times 10^{16}\,\mathrm{m} = 206,265\,\mathrm{AU}$
- energies: eV=electron volt=1.6×10^{-19} J (typical chemical reaction is eV; typical nuclear reaction is MeV)
- temperatures: often express as $k_B T$, where $k_B = 1.4 \times 10^{-23}$ J/K is Boltzmann's constant. $k_B T$ is an energy, can express in eV; 10^6 K is 86 eV
- Masses often expressed as energies (also in eV) via $E = mc^2$, so:
  - $m_e = 511\,\mathrm{keV}$ (electron)
  - $m_n \approx m_p \approx 1\,\mathrm{GeV}$ (neutron or proton)
  - $m_\gamma = 0$ (photon — rest mass)

And then we use usual metric-style prefixes to get things like kpc, Mpc, etc.

Google/Wolfram Alpha/astropy can be very helpful when checking unit conversions.

Lecture II Basic Concepts

Phillips Chapter 1. These are (in many cases) things we will go back over later.

II.2 Big Bang Nucleosynthesis

What ingredients do we have to make a star? Universe is mostly H, then He. Rest is details. How did it get that way?

It started out very hot (we know this since the Universe is expanding and we see the left-over radiation at 3 K now). Was a soup of interacting sub-atomic particles (electrons, positrons, neutrinos, quarks). Eventually (after 10^{-4} s) free quarks got bound up into neutrons, protons, ... Neutrons and protons in particular were in equilibrium:

$$\nu_e + n \rightarrow e^- + p$$

and

$$\bar{\nu}_e + p \rightarrow e^+ + n$$

at the same time. However, n is slightly more massive than p. At a temperature T, the ratio of these is given by the mass (energy) difference:

$$\frac{N_n}{N_p} = e^{-\Delta mc^2/k_B T}$$

This is a **Boltzmann factor** (will come back). Δmc^2 is energy difference, 1.3 MeV (remember that $m_p c^2 \approx 1$ GeV, so difference is 0.1%).

As T goes down and universe expands, the reactions go more slowly and we get more protons wrt neutrons. Finally it effectively stopped, and the ratio was frozen. This happened at $T \sim 10^{10}$ K, with $N_n/N_p \approx 1/5$. Then went down a little more (to 1/7) through natural decay of n.

At 10^9 K, could make deuteron:

$$n + p \rightarrow d + \gamma$$

From these could make ^{3}He, then ^{4}He. ^{4}He is very stable, so a lot of things got stuck there, except for a little ^{7}Li. But there were still a lot of protons left over. How much He?

$N_n/N_p \approx 1/7$. So take 2 neutrons, 14 protons (16 particles total, or a mass of ≈ 16 amu). Make a single ^{4}He nucleus, then 12 protons left. So out of 16 amu, 4 amu are in ^{4}He, or mass of He is $\approx 25\%$ total mass. This is pretty close to what we see.

II.3 Gravitational Contraction

Stars are one big fight against gravity. Temporary relief from thermonuclear fusion. But what are they fighting against?

Spherical system with M, R. Only have pressure, gravity. density is $\rho(r)$, pressure is $P(r)$.

Start at the center. How much mass out to r?

$$m(r) = \int_0^r dr'\rho(r')4\pi r'^2$$

($dr'\rho(r')4\pi r'^2$ is the mass of a shell at r'). Gravity only cares about enclosed mass, so:

$$g(r) = \frac{Gm(r)}{r^2}$$

What about pressure? Pressure on a parcel between r and $r + \Delta r$ (area=ΔA, volume=$\Delta r\Delta A$). If pressure at the top is the same as the bottom, no net force. But what if it is not the same?

$$P(r + \Delta r) \approx P(r) + \frac{dP}{dr}\Delta r$$

so difference (top $-$ bottom, or inward) in force (pressure times area) is:

$$\left[P(r) + \frac{dP}{dr}\Delta r - P(r)\right]\Delta A = \frac{dP}{dr}\Delta r\Delta A$$

But acceleration is force / mass, and mass is volume times density ($\Delta M = \rho(r)\Delta r\Delta A$). So acceleration from pressure is:

$$\frac{dP}{dr}\frac{1}{\rho(r)}$$

The total acceleration is then:

$$\frac{d^2r}{dt^2} = -g(r) - \frac{1}{\rho(r)}\frac{dP}{dr}$$

So if the star isn't moving, then P must increase toward the center ($dP/dr < 0$).

II.3.1 Free Fall

What if $P = 0$? Deal with energies, not acceleration. Convert potential energy to kinetic. Start at r_0, mass enclosed m_0. Initial $K = 0$, $U = -Gm_0^2/r_0$. $K + U$ is always the same, and $K = m_0v^2/2 = m_0(dr/dt)^2/2$. So:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_0}{r} = \frac{-Gm_0}{r_0}$$

Can get the time to go all the way to the center ($r = 0$):

$$t_{\rm FF} = \int_{r_0}^0 dr\frac{dt}{dr} = -\int_{r_0}^0 dr\left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2}$$

The integral is a little messy, but you can show that the free-fall time $t_{\rm FF}$is just:

$$\frac{\pi}{2}\left(\frac{r_0^3}{2Gm_0}\right)^{1/2}$$

Only depends on m_0/r_0^3. What has these units? $\rho = m_0/(4\pi r_0^3/3)$! So

$$t_{\rm FF} = \sqrt{\frac{3\pi}{32G\rho}}$$

For the Sun, $1/2$ hour. But for most things, eventually Pressure will stop collapse.

II.3.2 Hydrostatic Equilibrium

Assume 0 acceleration. Then:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} = -\rho(r)g(r)$$

This is a very important result: the equation of **hydrostatic equilibrium** (HSE). Applies to any stable system (atmospheres, stars, etc).

If the whole thing is in equilibrium at all r, then this will be true everywhere. Can then look at total potential energy:

$$\int_0^R dr\, 4\pi r^3 \frac{dP}{dr} = -\int_0^R dr\, \frac{Gm(r)\rho(r)4\pi r^3}{r^2}$$

where we multiplied both sides by $4\pi r^3$ and integrated. RHS is:

$$U_{\rm G} = -\int_0^M dm\, \frac{Gm(r)}{r}$$

where $dm = 4\pi r^2 \rho(r) dr$. Integrate LHS by parts:

$$P(r)4\pi r^3|_0^R - 3\int_0^R dr\, 4\pi r^2 P(r)$$

The first term is 0 ($P(R) = 0$). Second is average P times V: $\langle P\rangle V$. So:

$$\langle P\rangle = -\frac{U_G}{3V}$$

This is a very important result — one way of expressing the **virial theorem**. Can work for lots of things. What about particles in a box?

II.3.3 Kinetic Origin of Pressure

Box with side L has N particles. Particle hits top/bottom at a rate $v_z/2L$ (collisions/s) and imparts $2p_z$ (redirects with equal velocity). So momentum per time per area is $2p_z v_z/2L/L^2 = p_z v_z/L^3$. But momentum per time is force, and force per area is pressure. Total of N particles:

$$P = \frac{N}{L^3}\langle p_z v_z\rangle$$

Assume all directions are the same, so:

$$\langle p_x v_x\rangle = \langle p_y v_y\rangle = \langle p_z v_z\rangle = \langle \vec{p}\cdot\vec{v}\rangle/3 = \frac{n}{3}\langle \vec{p}\cdot\vec{v}\rangle/3$$

n is N/V or number density.

Can then do this for different types of particles. Total energy of a particle $\epsilon^2 = p^2c^2 + m^2c^4$ (kinetic + rest-mass). NR: $p \ll mc$, so $\epsilon = mc^2 + p^2/2m$. UR: $p \gg mc$ so $\epsilon = pc$. Can show:

$$P_{NR} = \frac{2}{3}\frac{K}{V}$$

$$P_{UR} = \frac{1}{3}\frac{K}{V}$$

So if NR:

$$\langle P \rangle = \frac{2}{3}\frac{K}{V} = -\frac{U_G}{3V}$$

or $U_G = -2K$. Other ways to write this are ($E = U + K$): $E = -K$, $E = U/2$. Overall $E < 0$ so the system is bound.

Strange consequence: add a little energy slowly. Add 1% of total energy, U goes down by 2%, K goes up by 1%. So for a contracting cloud converting gravitational energy to radiation (early model for the Sun) will get hotter (K goes up) as it contracts.

What if energy comes from nuclear reactions at the center? E goes up, so K goes down. Therefore it cools down! Adding energy makes it cooler?

II.4 Ideal Gas Law

You may know from chemistry:

$$PV = n_{\text{mol}}RT$$

with n_{mol} the # of moles, and R the ideal gas constant. But we are physicists. So the total number of molecules is $N_{\text{molec}} = N_A n_{\text{mol}}$, and we can write:

$$PV = N_{\text{molec}}\frac{R}{N_A}T$$

Divide both sides by V:

$$P = \frac{N_{\text{molec}}}{V}\frac{R}{N_A}T$$

Have $k_B = R/N_A$ is Boltzmann's constant. And:

$$n \equiv \frac{N_{\text{molec}}}{V}$$

is the **number density**: the number of particles per volume (units are m^{-3}, since number doesn't have a unit).

$$P = nk_BT$$

Can also write in terms of **mass density** ρ (kg m^{-3}):

$$\rho \equiv m_{\text{molec}}n$$

so

$$P = \frac{\rho}{m_{\text{molec}}}k_BT$$

II.5 Star Formation

Cloud collapses under influence of gravity. Details complicated. But basic conditions must be satisfied. Gravity must be stronger than pressure (kinetic energy).

$$U = -f\frac{GM^2}{R}$$

(f depends on density distribution, $f \sim 1$).

$$K = \frac{3}{2}Nk_BT$$

Need $|U| > K$ for collapse. Can write this as:

$$M > M_J = \frac{3k_BT}{2G\bar{m}}R$$

where $\bar{m}$ is average mass of particle ($M = N\bar{m}$). Or

$$\rho > \rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_BT}{2G\bar{m}}\right)^3$$

These are the **Jeans mass and density**.

So want a big cloud to collapse. But does a big cloud make a big star? Generally it breaks up along the way (*fragmentation*).

$T = 20\,\mathrm{K}$, $M = 10^3 M_\odot$, needs $\rho = 10^{-22}\,\mathrm{kg/m}^3$ ($n = 10^5\,\mathrm{m}^{-3}$) to collapse (not too bad). But for $1M_\odot$ density needs to be 10^6 times higher.

II.6 The Sun

$M = 1M_\odot$, $R = 1R_\odot$. So average density is $1.4 \times 10^3\,\mathrm{kg\,m}^{-3}$. $t_{\mathrm{FF}} = 30\,\mathrm{min}$, which isn't happening, so there must be pressure.

$$\langle P\rangle = \frac{-U}{3V} \approx \frac{1}{3}\frac{GM_\odot^2}{R_\odot}\frac{3}{4\pi R_\odot^3} = \frac{GM_\odot^2}{4\pi R_\odot^4} \approx 10^{14}\,\mathrm{Pa}$$

Can also say $\langle P\rangle = \langle\rho\rangle k_BT/\bar{m}$ (ideal gas law). $\bar{m} \approx 0.5\,\mathrm{amu}$ (ionized H). So

$$k_BT \approx \frac{GM_\odot\bar{m}}{3R_\odot} \approx 0.5\,\mathrm{keV}$$

or $T \approx 6 \times 10^6\,\mathrm{K}$. Hotter (and denser etc.) toward center.

II.6.1 What Powers the Sun and How Long Will It Last?

We take the Solar luminosity to be 4×10^{26} W, and try to find a way to get that amount of energy out over a long time.

The first estimate was due to Lord Kelvin (1862, in Macmillan's Magazine). This estimate (known now at the Kelvin-Helmholtz time, $t_{\rm KH}$) was shown to be < 100 Myr. But Darwin said (at the time) that fossils were at least 300 Myr old. So something weird was going on. Kelvin's estimate may have been wrong by a bit, but it couldn't be that bad. So there had to be some unknown energy source.

The lifespan of the Sun could be due to:

1. Chemical energy
2. Gravitational energy
3. Thermal energy (could it have just been a lot hotter in the past?)
4. Fission?

The answers for all of these are no. Kelvin's estimate concerned specifically gravitational. Chemical energy isn't enough, since we know about how much chemical energy a given reaction can release for a given amount of stuff. Same with fission.

II.6.2 Gravito-Thermal Collapse, or the Kelvin-Helmholtz Timescale

This ascribes the luminosity to the change in total energy: L is change in $E = K + U$.

If you do this you get a timescale of $t_{\rm KH} \sim 10^7$ yr, which is $\gg t_{\rm ff}$:

$$t_{\rm KH} \sim \frac{E}{L}$$

But $E \sim GM_\odot^2/R_\odot \sim 10^{41}\ \mathrm{J} = 10^{48}$ erg (1 J=10^7 erg).

That is because as collapse occurs, $|U|$ increases so K increases too. That heats up the star, which slows down the collapse.

We can use the Virial theorem to get the central temperature T_c of the Sun. We assume that the center (the hottest/densest bit) dominates K:

$$K \sim \frac{3}{2} k_B T_c \frac{M}{\bar{m}}$$

with $\bar{m} \approx m_p$ the average particle mass. And $K = -U/2$, with $U \sim -GM_\odot^2/R_\odot$. So we find $T_c \sim GM_\odot m_H/k_B R_\odot \sim 10^7$ K. This is pretty good (the real number is about 1.6×10^7 K).

II.6.3 Solar Radiation

Stars are close to blackbodies. Define effective temperature:

$$L = 4\pi R^2 \sigma T_{\rm Eff}^4$$

σ is the Stefan-Boltzmann constant. For the Sun, $T_{\rm Eff} \approx 6000\,{\rm K}$. This means the blackbody peaks in the visible portion of the spectrum. And this is much cooler than the interior.

How does it get from very hot interior to cool exterior?

Center of the Sun: nuclear reaction releases energy in the form of neutrinos (which escape) and photons (gamma-rays). How long to get out? A naive answer is $\sim R_\odot/c = 2\,{\rm s}$. But not for photons.

It actually takes $\sim 10^7$ yrs. Why? Because a star is a very crowded place, and photons (even though they move fast) cannot move very far before they wack into something else and end up going in another direction. They easily bounce (scatter) off of ions, electrons, and atoms, and even other photons.

Each bounce tends to make the photon lose energy, but more photons are then produced, conserving energy. In the center the photons start out as X-ray photons, but by the time they get to the surface of the star they are optical photons. They get there via a *random walk*.

Assume that a photon will move (on average) a distance $l_{\rm mfp}$ before it hits something and changes direction. That distance is the *mean free path*. It travels a distance d after N collisions. We can determine what $d(N)$ is. Assume each one moves $\vec{l}_i$ for $i = 1 \ldots N$, with $|\vec{l}_i| = l_{\rm mfp}$. So the total distance is the vector sum:

$$\vec{d} = \sum^N \vec{l}_i$$

We want the magnitude of this, $|\vec{d}| = \sqrt{\vec{d}\cdot\vec{d}}$. But

$$\vec{d}\cdot\vec{d} = \sum_i^N \vec{l}_i \cdot \vec{l}_i + \sum_{i\neq j} \vec{l}_i \vec{l}_j$$

The second term there will go to 0 on average, since the directions are different. So $|\vec{d}|^2 = N|\vec{l}| = N l_{\rm mfp}$, or $d = \sqrt{N} l_{\rm mfp}$. This is in fact a general result with applicability to a wide range of areas.

From this we can determine how long does it take for a photon to diffuse out of the star. To go a distance d, it takes:

$$\begin{array}{ccl} & \frac{d}{c} & l_{\rm mfp} > d \\ N\frac{l_{\rm mfp}}{c} = & \frac{d^2}{l_{\rm mfp} c} & l_{\rm mfp} < d \end{array}$$

This is also often referred to as a "drunkard's walk". So to go $R_\odot$ it takes:

$$\frac{R_\odot^2}{lc}$$

which is a factor of $R_\odot/l$ longer than basic escape. So luminosity (energy per time) also changes by that factor. Naive luminosity for central temperature is:

$$L = 4\pi R_\odot^2 \sigma T_I^4$$

but in reality it is $T_{\rm Eff} = 6000\,{\rm K} \ll T_I = 6{,}000{,}000\,{\rm K}$. So:

$$T_{\rm Eff} \approx T_I \left(\frac{l}{R_\odot}\right)^{1/4}$$

which would give $l \sim 1\,{\rm mm}$ (very small!). Which would give about 50,000 yr to diffuse (too small, but not horrible).

II.7 Stellar Life Cycles

Big Bang: mostly H and He. Stars make the rest. T at the center of a star is pretty close to constant, set by fusion (hotter $\rightarrow$ faster $\rightarrow$ bigger $\rightarrow$ cooler). So $M/R \sim$ constant.

Energy escape determines luminosity. Since $L \sim R^2 T_I^4 (l/R) \sim R^2 (M/R)^4 (l/R) \sim \rho M^3$.

Since $L \sim M^3$ (roughly), determined by how fast energy can escape. So lifetime is $\sim M/L \sim M^{-2}$: bigger stars use up their fuel much faster. About 10^{10} yr for the Sun.

II.8 Color-Magnitude Diagram

Plot $T_{\rm Eff}$ increasing to the left, L increasing up. Hotter is the same as bluer, so often plot color (blue to the left) on the x-axis. We can directly observe color. And instead of L plot magnitude, where $m = m_\odot - 2.5 \log_{10}(L/L_\odot)$. So it decreases going up, but that still means brighter.

Most of the stars define the **Main Sequence**. This turns out to be where normal H fusion is occuring. Can also identify regions for **Red Giants** and **White Dwarfs**.

Lecture III Matter & Radiation

Phillips, Chapter 2

Star: matter and radiation fighting against collapse. Material can be extreme (ionized, degenerate, relativistic, ...) but can gain insight from ideal gases. We start with a pretty general framework, and then work to specific examples.

III.2 Ideal Gas

Energy of quantum states of particles not affected by interactions. Could be atoms, molecules, ions, electrons, photons, etc.

III.2.1 Density of States

Look at wave-like properties. Each particle in a box with side L. So each particle has a standing wave with wave vector (k_x, k_y, k_z) such that integer number of cycles in L: $k_x = n_x\pi/L$ etc. Don't care about direction, only magnitude k. Look at all of the states in a spherical shell between k and $k + dk$. Volume of this is $4\pi k^2\, dk/8$ (only 1 octant of sphere since $k > 0$). Number of states is:

$$\left(\frac{L}{\pi}\right)^3 \frac{4\pi k^2\, dk}{8}$$

We want to look at momenta. $p = h/\lambda$ (λ is de Broglie wavelength), or $p = \hbar k$. So number of states in a momentum bin between p and $p + dp$ is:

$$g(p)dp = \frac{V}{h^3} 4\pi p^2\, dp$$

If there is also spin, multiply by g_s (number of spin states or polarizations). $g_s = 2$ for electrons, photons.

III.2.2 Internal Energy

Density of states + energy of states + number of particles per state give internal energy. General energy of states:

$$\epsilon^2 = p^2c^2 + m^2c^4$$

So total energy is:

$$E = \int_0^\infty dp\, g(p)\epsilon(p)f(\epsilon)$$

where $f(\epsilon)$ is average number of particles with energy ϵ. Total number of particles is:

$$N = \int_0^\infty dp\, g(p)f(\epsilon)$$

Thermodynamics: temperature T, pressure P, chemical potential μ:

$$dE = TdS - PdV + \mu dN$$

gives changes in total energy from changes in entropy S, volume V, number of particles N. But still need to know $f(\epsilon)$.

If fermions (Pauli exclusion principle: only 1 particle per state):

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/k_B T] + 1}$$

This works for electrons, atoms.

If bosons (can be many particles per state):

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/k_B T] - 1}$$

This works for photons. Bosons can have many more particles at low ϵ.

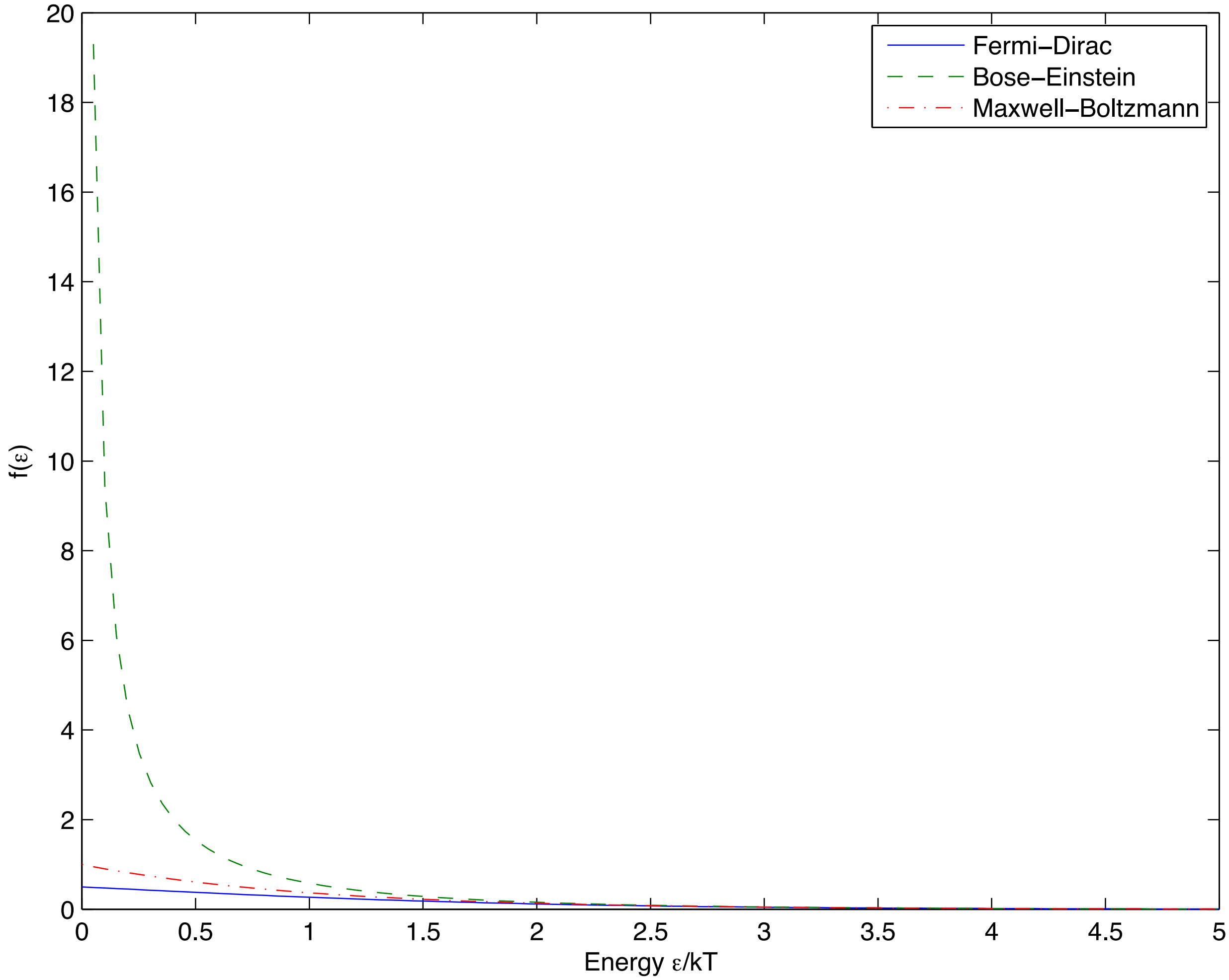
As T increases density of states decreases, and fermions and bosons both look similar → dilute classical gas. This happens when:

$$e^{(mc^2 - \mu)/k_B T} \gg 1$$

so both become:

$$f(\epsilon) \approx e^{-(\epsilon - \mu)/k_B T} \ll 1$$

All states have $\ll 1$ particle, so Pauli exclusion does not matter. Instead everything just follows Maxwell-Boltzmann statistics.


Fermi–Dirac
Bose–Einstein
Maxwell–Boltzmann
f(ε)
Energy ε/kT
0
2
4
6
8
10
12
14
16
18
20
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

III.2.3 Pressure

Like we did before, relate pressure to internal energy (did for classical — quantum here).

Change volume by dV, hold others fixed. So $dE = -PdV$. Therefore

$$P = -\frac{\partial E}{\partial V} = -\int_0^\infty \frac{d\epsilon}{dV} f(\epsilon) g(p) dp$$

From this can find the same relation as before for $P(E)$.

III.2.4 Ideal Classical Gas

Generally familiar, but look at relativistic and when classical parts fail.

When occupation $\ll 1$:

$$P = \frac{1}{3V} e^{\mu/k_B T} \int_0^\infty p v_p e^{-\epsilon/k_B T} g_s \frac{V}{h^3} 4\pi p^2 dp$$

But $d\epsilon = vdp$, so integral is

$$\int_0^\infty p^3 e^{-\epsilon/k_B T} v_p dp = -k_B T \int p^3 d(e^{-\epsilon/k_B T})$$

Integrate by parts, get:

$$P = \frac{k_B T}{V} e^{\mu/k_B T} \int_0^\infty e^{-\epsilon/k_B T} g_s \frac{V}{h^3} 4\pi p^2 dp$$

But the integral is just like the integral that we do for N, so

$$P = \frac{N}{V} k_B T = n k_B T$$

Ideal Gas Law!.

If we want total number of particles, can use $\epsilon = mc^2 + p^2/2m$ for non-relativistic particles to get:

$$N = e^{(\mu - mc^2)/k_B T} g_s \frac{V}{h^3} (2\pi m k_B T)^{3/2}$$

Or can solve:

$$\mu - mc^2 = -k_B T \ln\left(\frac{g_s n_Q}{n}\right)$$

where $n = N/V$ and :

$$n_Q \equiv \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

is the **quantum concentration**.

If UR, then:

$$\mu = -k_B T \ln\left(\frac{g_s n_Q}{n}\right)$$

and

$$n_Q \equiv 8\pi \left(\frac{k_B T}{hc}\right)^3$$

What do these mean? Remember that classical if concentration is $\ll 1$. This happens when

$$e^{(mc^2-\mu)/k_B T} \gg 1$$

or if $n \ll n_Q$. If there is $\ll 1$ particle per quantum "box", then classical. Size of box is:

$$\lambda = h/p \approx h/\sqrt{mk_B T}$$

(for NR), and separation between particles is $\sim n^{-1/3}$. For UR, $\lambda \approx hc/k_B T$. Want separation to be $\gg \lambda$.

Photons are UR, but also have $\mu = 0$, $m = 0$, so are always quantum. But if $m > 0$ can be quantum or classical.

Difference happens fastest for lightest particles, since λ increases as m decreases. So when we make stuff denser (star contracts), electrons become quantum first. This leads to many important effects.

III.3 Electrons in Stars

Here relativistic and quantum effects can both be important. But are they important in the Sun? $\rho \sim 10^3\,\mathrm{kg/m^3}$, $T \sim 6 \times 10^6$ K.

Compare $k_B T$ to $m_e c^2$: $k_B T \approx 10^{-3} m_e c^2$, so pretty NR.

Compare n to n_Q: $n \approx 6 \times 10^{29}\,\mathrm{m^{-3}}$ (based on ρ). $n_Q \approx 3 \times 10^{31}\,\mathrm{m^{-3}}$ for $T = 6 \times 10^6$ K, so pretty ideal too ($n \ll n_Q$).

But what will happen as the Sun evolves and contracts? $T \sim M/R$, so as R goes down T goes up, and as T goes up n_Q goes up.

$$n_Q \sim T^{3/2} \sim R^{-3/2}$$

But n also goes up, and it goes up faster:

$$n \sim R^{-3}$$

So from this n will eventually exceed n_Q, and quantum effects will become important.

III.4 Degenerate Electrons

Very important concept

Quantum effects, $n \gg n_Q$, means high density. *OR* low temperature

$$k_B T \ll \frac{h^2 n^{2/3}}{2\pi m_e}$$

So it can still be $> 10^6$ K, but cold in a quantum sense. This is a degenerate gas: quantum states overlap, and electrons end up piled into the states with the lowest energies. Pauli exclusion principle is very important. Take the $f(\epsilon)$ from before. If $\mu = 0$ and $T \to 0$, then:

$$f(\epsilon) = \begin{cases} 1 & \epsilon \leq \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases}$$

ϵ_F is the **Fermi Energy**: all states up to there are filled, and no states are above there. Can also have Fermi momentum p_F.

Can then get the total number from the distribution (easy to integrate now):

$$N = \int_0^{p_F} g_s \frac{V}{h^3} 4\pi p^2 dp = \frac{8\pi V}{3h^3} p_F^3$$

Since we can use $n = N/V$:

$$p_F = \left(\frac{3n}{8\pi}\right)^{1/3} h$$

since $p = h/\lambda$, $\lambda \sim n^{-1/3}$ as we would expect (each particle occupies the smallest possible quantum box).

We can get the other properties (pressure etc). NR is $p_F \ll m_e c$, or $n \ll (mc/h)^3$ where $h/mc = 2.4 \times 10^{-12}$ m is the Compton wavelength of electron. So $\epsilon = m_e c^2 + p^2/2m_e$ and

$$E = N\left(m_e c^2 + \frac{3p_F^2}{10m_e}\right)$$

Since $P = 2K/3V$ for a NR gas, identify the second term as K and say:

$$P = n\frac{p_F^2}{5m_e}$$

Substitute back in and get:

$$P = K_{NR} n^{5/3}$$

with

$$K_{NR} = \frac{h^2}{5m_e}\left(\frac{3}{8\pi}\right)^{2/3}$$

This general result $P \propto n^{5/3}$ is very important. For UR, $n \gg (m_e c/h)^3$ and $\epsilon = pc$ so:

$$E = \frac{3}{4} N p_F c$$

which gives $P = np_F c/4$, or

$$P = K_{UR} n^{4/3}$$

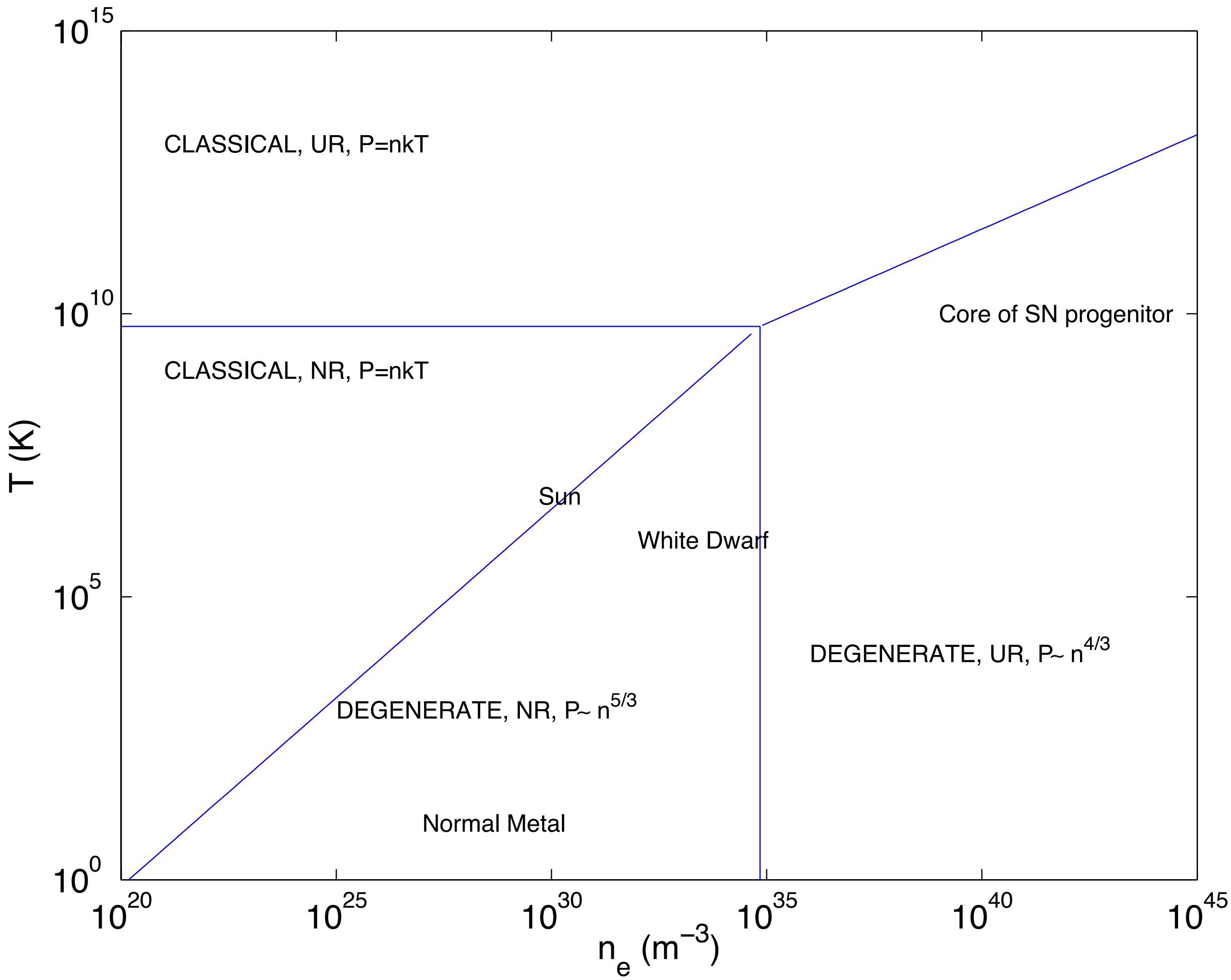
with

$$K_{UR} = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}$$

Now $P \propto n^{4/3}$. This change in exponent has important consequences.

III.4.1 Density Temperature Diagram

Different combinations of n, T give different results. Classical vs. degenerate, NR vs. UR. Compare n to n_Q in UR and NR limits, and for $n \ll n_Q$ compare $k_B T$ to $m_e c^2$ (Figure 2.2).



Dividing lines $kT = m_e c^2$ or $T = 6 \times 10^9$ K, $n \sim 7 \times 10^{34}/\text{m}^3$

When are things not ideal? When direct interactions occur, mostly electrostatic (electrons + ions). This is important when $Ze^2/4\pi\epsilon_0 r \sim k_B T$, or:

$$\frac{Ze^2}{4\pi\epsilon_0 k_B T} n^{1/3} > 1$$

But this only works if $\epsilon \sim k_B T$, which does not apply if degenerate. Then ϵ is only a function of the density, so (NR) look at:

$$\frac{Ze^2 2m_e}{4\pi\epsilon_0 h^2} n^{-1/3}$$

As n increases this gets lower, so electrostatics become less important.

III.4.2 Electrons in the Sun

At the center, $n_e \sim 8 \times 10^{31}\,\mathrm{m}^{-3}$, $T = 1.6 \times 10^7$ K. So $n_Q = 1.5 \times 10^{32}\,\mathrm{m}^{-3}$ and we are not that far off. Mostly classical, but need corrections.

Sun burns H, turns into He. Center will turn into He with burning H around that, T will increase to 2×10^7 K, n_e up to $3 \times 10^{32}\,\mathrm{m}^{-3}$. This is above n_Q: center becomes degenerate.

Eventually $n_e \sim 10 n_Q$, so "cold" gas at 10^8 K.

But soon will become hot enough for He to burn (into C). If were non-degenerate, fusion makes energy, makes gas expand, makes temperature fall, so fusion slows down (regulated). But not if degenerate. Then temperature will not change during expansion, so will not cool down. He burning goes uncontrolled in a **helium flash**. Lots of burning, but most of the energy does not escape as radiation — instead it expands and eventually becomes non-degenerate. Finally you get normal, controlled fusion.

Sun cannot go past this. Will leave behind **white dwarf** composed of mostly C and O. Central density will be $\sim 10^9\,\mathrm{kg\,m^{-3}}$ for $0.5 M_\odot$, 10^8 K. Ions will cool gradually, but electrons hold up the star through denegeracy pressure.

III.4.3 Electrons in Massive Stars

Degeneracy is less important:

$$k_B T \approx \frac{GM\bar{m}}{3R} \propto M^{2/3} \rho^{1/3}$$

Need a given T to ignite fusion. But for higher M this will happen at a lower ρ, so massive stars are less degenerate. Some stars ($> 11 M_\odot$) can evolve and burn all the way to Fe without degeneracy worries.

But this will matter at the end. When fusion stops, density increases, and electrons will be ultra-relativistic & degenerate: $T \approx 8 \times 10^9$ K, $\rho \approx 4 \times 10^{12}\,\mathrm{kg\,m^{-3}}$. So $n_e \approx 10^{39}\,\mathrm{m}^{-3} \gg n_Q$, and $p_F \approx 12mc$. This leads to collapse of the central core $\rightarrow$ supernova.

III.4.4 Stability and the Adiabatic Index

(Phillips 1.2)

Look at generic equation-of-state of a gas. Adiabat: entropy is constant, given by PV^γ=constant with γ the adiabatic index. (related to degrees of freedom, $\gamma = 1 + 2/\mathrm{DOF}$: for a non-relativistic monatomic gas with only 3 DOF, $\gamma = 5/3$).

If PV^γ=constant:

$$\gamma \frac{dV}{V} + \frac{dP}{P} = 0$$

or

$$d(PV) = PdV + VdP = -(\gamma - 1) PdV$$

Since this is adiabatic, no heat is added: change in internal energy is just from the work done. So

$$dK = -PdV = \frac{1}{\gamma - 1} d(PV)$$

If γ is a constant we can integrate and get:

$$K = \frac{1}{\gamma - 1} PV$$

What if we have a gravitating bound gas in equilibrium. Then use virial theorem:

$$\langle P \rangle = (\gamma - 1)\frac{K}{V} = -\frac{1}{3}\frac{U}{V}$$

So we have

$$3(\gamma - 1)K + U = 0$$

What we had derived before for NR and UR gases works if $\gamma_{NR} = 5/3$ and $\gamma_{UR} = 4/3$.

Total energy

$$E = K + U = -(3\gamma - 4)K$$

To be bound, we need $E < 0$, so this requires $\gamma > 4/3$. As $\gamma \to 4/3$ then E approaches 0 and the gas becomes very loosely bound, and very small changes in the total energy come with very large changes in K and U. This becomes unstable, as small perturbations can drive $E \geq 0$. And this is relevant, since for a NR gas $\gamma = 5/3$ and things are stable, but as the gas becomes UR and $\gamma \to 5/3$ the gas becomes unstable. Other effects (new ways of absorbing heat, such as ionization) will have the same result.

What this means is that changes in energy almost cancel exactly between K and U. For instance: $\gamma = 4/3 \cdot 1.01$. Add 25 J to K, U will become more negative by 26 J, so E will become more negative only by 1 J. Anything that throws off the balance will mess this up.

Relation to electrons: $P_{NR} \propto n^{5/3}$, $P_{UR} \propto n^{4/3}$. But $n = 1/V$, so we have $P_{NR} \propto (1/V)^{5/3}$ or $P_{NR}V^{5/3}$ =constant which says that $\gamma = 5/3$ for NR electrons (as I said above) and $4/3$ for UR electrons. So as electrons become UR, the star gets unstable.

III.5 Photons in Stars

Electrons play a major role in stars. They carry a lot of the pressure, they become degenerate, they conduct, and they scatter. Aside from the ions the other major constituent are the photons (radiation). This can also be an important source of pressure and transport of energy.

III.5.1 The Photon Gas

EM radiation: assume a blackbody, or an ideal gas of photons (must be quantum since $m = \mu = 0$). All particles move at c. But since can make or remove photons at will, $\mu = 0$.

$$dE = TdS - PdV + \mu dN$$

N will change to maximize S at fixed E, V. So

$$\frac{\partial S}{\partial N} = -\frac{\mu}{T} = 0$$

Can show with a bit more thermo that this happens when $\mu = 0$.

For photons, get number per state:

$$N(p)dp = \frac{1}{e^{\epsilon/k_BT} - 1} g_s \frac{V}{h^3} 4\pi p^2\, dp$$

With $\epsilon = pc$ (photons are relativistic) and $g_s = 2$. So integrate over all energies to get total number density:

$$n = \frac{1}{V}\int_0^\infty N(p)dp$$

The math is slightly messy (see book), but we get $n = bT^3$ with $b = 2.404(8\pi k_B^3/h^3c^3) = 2.03 \times 10^7\,\mathrm{K}^{-3}\,\mathrm{m}^{-3}$. And can do the same for E to get energy density $u = E/V$, finding $u = aT^4$ with $a = 8\pi^5 k_B^4/15h^3c^3 = 7.565 \times 10^{-16}\,\mathrm{J\,K}^{-4}\,\mathrm{m}^{-3}$. Or you can combine to get $u = 2.70nk_BT$. Which means that the average energy of a photon is $2.70k_BT$. Compare to $1.5k_BT$ for a classical gas and $3k_BT$ for a UR gas.

Just like with electrons, the pressure is related to the energy density:

$$P = \frac{u}{3} = \frac{1}{3}aT^4$$

You can also think of a box with a small hole through which photons escape. Look at the amount of energy escaping through that hole. Photons escape at a rate of $nc/4$ per unit area: effective speed is $c/4$ comes from integrating photons moving in all directions. [In all directions, photons spread out over 4π steradians

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi$$

where $d\Omega = \sin\theta d\theta d\phi$ is a spherical angle element. Going through the whole, we multiply the photons transport of energy by $\cos\theta$ since this is the projection of their motion onto the perpendicular, and only integrate for half of the sphere $\theta = 0 \to \pi/2$. Which gives us π, and this is $1/4$ of the total from before.]

So the rate of energy escape is $uc/4$ per unit area, or $acT^4/4$. We say:

$$F = \sigma T^4\,\mathrm{W\,m}^{-2}$$

is the energy flux, with $\sigma = ac/4 = 5.67 \times 10^{-8}\,\mathrm{W\,K}^{-4}\,\mathrm{m}^{-2}$ the **Stefan-Boltzmann constant**.

We can also write this in terms of intensity I_ν: J/s/m^2/Hz, which is energy per time per area per frequency interval:

$$I_\nu d\nu = \frac{c}{4}u_\nu d\nu = \frac{c}{4}\frac{h\nu}{e^{h\nu/k_BT} - 1}\frac{4\pi\nu^2}{c^3}d\nu$$

Which is the Planck blackbody function.

You can differentiate this wrt ν and find a maximum at $\nu = 2.82 k_B T/h$, which says that the most probable energy is $2.82 k_B T$. This is slightly different from the average energy (mean vs. mode).

III.5.2 Radiation Pressure

How important is it in stars? Look at solar surface (photosphere) and interior.

Property	Surface (6×10^3 K)	Interior (6×10^6 K)
Average photon energy (eV)	1.4	1400
Photon density n (m^{-3})	4×10^{18}	4×10^{27}
Radiation energy density u ($\mathrm{J\,m}^{-3}$)	1	1×10^{12}
Radiation pressure (Pa)	0.33	0.33×10^{12}
Radiation intensity σT^4 ($\mathrm{MW\,m}^{-2}$)	73	73×10^{12}

At the surface, radiation pressure $P_{\rm rad}$ is tiny, even compared with pressure on the Earth (1 atm = 10^{-5} Pa). But much more significant inside. Even so, $\ll 10^{14}$ Pa needed to keep the star from collapsing. So the Sun is mostly supported by pressure from ions and electrons.

What about other stars? $T \sim M/R$, and $\rho \sim M/R^3$. So

$$P_g = n_e k_B T + n_i k_B T \propto \frac{M^2}{R^4}$$

(as we said before). What about radiation?

$$P_r = \frac{a}{c} T^4 \propto \frac{M^4}{R^4}$$

So

$$\frac{P_r}{P_g} \propto M^2$$

There can be stars that are massive enough that radiation pressure dominates. In particular, for $M > 50\,M_\odot$ this matters a lot. And this messes things up since photons are relativistic: they have an adiabatic index of $4/3$ so stars supported by P_r are not very stable.

III.6 The Saha Equation

Basic way to keep track of populations of related states in equilibrium. For instance, molecules $\leftrightarrow$ atoms, or atoms $\leftrightarrow$ ions+electrons.

Consider H in equilibrium with radiation, and derive Saha equation for ionization of H.

Need to think about chemical potential μ. If one type of particle, they move from high μ to low μ until μ is the same everywhere (make energy the same, and $dE = \mu dN$, so if μ is high make $dN < 0$). What if we have multiple types of particles? Consider A, B, C, D:

$$A + B \leftrightarrow C + D$$

In equilibrium where chemical potential is equal on both sides:

$$\mu(A) + \mu(B) = \mu(C) + \mu(D)$$

H atoms: electrons are in bound states with $\epsilon_n = -13.6\,\mathrm{eV}/n^2$ and $n = 1, 2, \ldots$ (Bohr atom). When it is ionized, electron can have any momentum $\vec{p}$ and energy $\epsilon(p) = p^2/2m_e$ (NR).

Equilibrium: photon hits atom, liberates electron. Eventually electron hits ion, gets captured, releases photon. So reaction is:

$$\gamma + \mathrm{H}_n \leftrightarrow e^- + p$$

(p here is proton). Since $\mu = 0$ for photon:

$$\mu(\mathrm{H}_n) = \mu(e) + \mu(p)$$

Treat as classical ideal particles, NR:

$$\mu(e) = m_e c^2 - k_B T \ln\left(\frac{g_e n_{Q,e}}{n_e}\right)$$

$$\mu(p) = m_p c^2 - k_B T \ln\left(\frac{g_p n_{Q,p}}{n_p}\right)$$

$$\mu(\mathrm{H}_n) = m_{\mathrm{H,n}} c^2 - k_B T \ln\left(\frac{g(\mathrm{H}_n) n_{Q,p}}{n_{\mathrm{H}}}\right)$$

We know the mass of the bound atom:

$$m(\mathrm{H}_n)c^2 = m_e c^2 + m_p c^2 + \epsilon_n$$

(mass of electron + proton - binding energy). n_Q for H is basically that for proton, since depends on the de Broglie wavelength. And $g_e = g_p = 2$. For the bound atom the degeneracy is different. $g(\mathrm{H}_n) = 4n^2$ counting electron, proton, and angular momentum states. So we can find:

$$\frac{n(\mathrm{H}_n)}{n_e n_p} = \frac{g_n}{n_{Q,e}} e^{-\epsilon_n/k_B T}$$

We then consider all possible bound states, and sum up over n. So

$$\frac{n(\mathrm{H})}{n_e n_p} = \frac{1}{n_{Q,e}} \sum_{n=1}^{\infty} g_n e^{-\epsilon_n/k_B T}$$

Or:

$$\frac{n(\mathrm{H})}{n_e n_p} = \frac{Z}{n_{Q,e}} e^{E_i/k_B T}$$

with $E_i = -\epsilon_1 = 13.6\,\mathrm{eV}$ is the ionization energy ($n = 1 \to \infty$) and

$$Z \equiv \sum_{n=1}^{\infty} g_n e^{-(\epsilon_n - \epsilon_1)/k_B T}$$

The energy difference is the excitation energy of the nth level. Z is a partition function: it tells how many particles are in each sub-state. It can formally diverge, but in reality this would be for an infinitely large atom. So we truncate the sum where the size of the atom becomes comparable to the distance between atoms. We end up with $Z \approx 1$.

With that, we can write:

$$\frac{n(\mathrm{H}^+)}{n(\mathrm{H})} \approx \frac{n_{Q,e}}{n_e} e^{-E_i/k_B T}$$

So a very strong temperature dependence. As T goes up, we get many more ions. But also depends on density: if density decreases, ionization also goes up (once it is ionized, will stay ionized longer).

III.7 Ionization in Stars

III.7.1 Stellar Interiors

First only H. So we have ions and atoms. Total mass density:

$$\rho = m_{\mathrm{H}}(n(\mathrm{H}) + n(\mathrm{H}^+))$$

We then write $x(\mathrm{H}) = n(\mathrm{H}^+)/n(\mathrm{total})$ is the ionization fraction. So:

$$n_e = n(\mathrm{H}^+) = x(\mathrm{H})\rho/m_{\mathrm{H}}$$

and

$$n(\mathrm{H}) = (1 - x(\mathrm{H}))\rho/m_{\mathrm{H}}$$

We can then solve for x using the Saha equation and find:

$$\frac{1 - x(\mathrm{H})}{x(\mathrm{H})^2} \approx \frac{\rho/m_{\mathrm{H}}}{10^{21}T^{3/2}} e^{158{,}000/T}$$

So for the average conditions we had been assuming for the Sun, $\rho \approx 1400\,\mathrm{kg\,m^{-3}}$ and 6×10^6 K, we get $x(\mathrm{H}) \approx 95\%$. Which says that H is mostly but not completely ionized. However, this is an underestimate. The H atoms are not an ideal gas at this density. The typical spacing between particles is $\sim (\rho/m_{\mathrm{H}})^{-1/3} = 10^{-10}\,\mathrm{m}$ or 1 Å, which is the typical size of an atom. So atoms interact strongly (**not an ideal gas**) and the amount of ionization is increased.

Heavy elements (there are some)? The inner electrons will be very tightly bound, but the number of total electrons (provided by ionized H) is so high that it surrounds the small number of atoms and keeps them ionized.

Example: C in H at $\rho \approx 1400\,\mathrm{kg\,m^{-3}}$ and 6×10^6 K. All H ionized, so $n_e = 8 \times 10^{29}\,\mathrm{m^{-3}}$. Carbon has $Z = 6$ protons, so the ionization energy is Z^2 times that of H for the inner-most electron. We look for ions with 1 electron left compared to those with 0:

$$\frac{n(\mathrm{C}^{+6})}{n(\mathrm{C}^{+5})} \approx \frac{10^{21}T^{3/2}}{n_e} e^{-36\times 158{,}000/T} \approx 10$$

Again this is a bit of an underestimate, but it suggests that things are pretty ionized.

If we **assume** everything is completely ionized, then it simplifies further analysis. $X_{\rm H}$, $X_{\rm He}$, X_A are the mass fractions of H, He, metals. When ionized, we can determine the total number density (which matters for pressure, e.g.):

$$n \approx 2n_{\rm H} + 3n_{\rm He} + \frac{A}{2}n_A = \left(2X_{\rm H} + \frac{3}{4}X_{\rm He} + \frac{1}{2}X_A\right)\frac{\rho}{m_{\rm H}}$$

where we assume $Z \approx A/2$ electrons per neutron in the heavy elements. Since the mass fractions add to 1, we can say:

$$n \approx \left(1 + 3X_{\rm H} + \frac{1}{2}X_{\rm He}\right)\frac{\rho}{2m_{\rm H}}$$

From which we can get the average particle mass $\bar{m} = \rho/n$. In the standard Solar model, $X_{\rm H} = 0.71$, $X_{\rm He} = 0.27$, and $X_A = 0.02$, so $\bar{m} \approx 0.61$ amu. However, the fractions change with depth, so deep inside $\bar{m} \approx 0.85$ amu.

III.7.2 Stellar Atmospheres

These are the portions that give rise to the spectra we actually see (where $T = T_{\rm Eff}$). For stars, go from 30,000 K to 3,000 K. As this happens the spectra change considerably, going from lines of He, to H, to metals, to molecules. That is mostly just from T effects (not changes in composition). Look at importance of ionization.

Metals (Li, Na, Mg, Al, K, Ca) has outer electrons that are weakly bound, ~ 5 eV. For $T \geq 5000$ K (0.4 eV) they are mostly ionized. Other elements (H, C, N, O) are typically partially ionized (more at high T). He, Ne are very tightly bound (~ 20 eV): ionized only at highest T.

Use H, He, Na to demonstrate, with binding energies of 13.6 eV, 24.6 eV, 5.14 eV.

$$\begin{aligned}
\frac{n({\rm Na}^+)}{n({\rm Na})} &\approx \frac{10^{21}T^{3/2}}{n_e}e^{-60{,}000/T} \\
\frac{n({\rm H}^+)}{n({\rm H})} &\approx \frac{10^{21}T^{3/2}}{n_e}e^{-158{,}000/T} \\
\frac{n({\rm He}^+)}{n({\rm He})} &\approx \frac{10^{21}T^{3/2}}{n_e}e^{-286{,}000/T}
\end{aligned}$$

So at a typical temperature of 6,000 K

$$\frac{n({\rm Na}^+)}{n({\rm Na})} \approx 10^7\frac{n({\rm H}^+)}{n({\rm H})}$$

$$\frac{n({\rm He}^+)}{n({\rm He})} \approx 10^{-10}\frac{n({\rm H}^+)}{n({\rm H})}$$

Very dramatic differences! So even though Na is 10^{-6} of H by number, it supplies more electrons. For a typical density, 10^{-4} of H is ionized at 6000 K, but 50% is ionized at 9000 K, and 50% of He at 15,500 K.

III.7.3 Spectral Classification

Depend on differences in spectral lines, such as presence of absence of Balmer ($n = 2$) lines of H. If we see Balmer lines, there must be atoms with electrons in the $n = 2$ state. If it were too cold then all electrons would be in $n = 1$. If too hot, all ions. We find this happens for 6000–11,000 K.

Hotter stars: might have He lines. From He^+, so lost single electron. This happens between 12,000–30,000 K. Metal lines are for cooler stars (where they are not totally ionized), 3000–6000 K.

Overall, atmospheres obey Kirchoff's laws. The bottom parts are hot and opaque and emit at all wavelengths. Outer lays let most wavelengths pass through except for a few that correspond to transitions of interest, so we get absorption lines. But there is another source of absorption at a range of wavelengths coming from H^- ions.

This is one proton + 2 electrons. Weakly bound, 0.75 eV. So a long-wavelength photon can knock away an electron:

$$\gamma + \mathrm{H}^- \leftrightarrow e^- + \mathrm{H}$$

This can happen for photon energies down to 0.75 eV, or wavelengths out to 1650 nm (near-IR). When this happens, photons of many wavelengths are all absorbed, and the material becomes opaque.

But we need electrons! From where? A small amount of metals M.

$$n_e = n(\mathrm{M}^+) = x(\mathrm{M})(n(\mathrm{M}) + n(\mathrm{M}^+))$$

with ionization fraction x. If similar to sodium, use Saha equation:

$$\frac{1-x}{x^2} \approx \frac{n(\mathrm{M}) + n(\mathrm{M}^+)}{10^{21}T^{3/2}} e^{60{,}000/T}$$

And then we can write:

$$\frac{n(\mathrm{H}^-)}{n(\mathrm{H})} \approx \frac{n_e}{10^{21}T^{3/2}} e^{8700/T}$$

We can solve these to get the ionized metal fraction and the concentration of H^- ions. We find the H^- ions peak around 3500 K and go down on either side. On one side there are not enough electrons, while on the other things are too energetic to stay bound. So if $T < 3000\,\mathrm{K}$ it will no longer absorp visible radiation. This means that the layer of gas we see (which has to absorb radiation, since otherwise it would cool way down) is at about 3000 K for most stars.

This matters for red giants. L increases. As that happens the outer part of the star moves out, so R increases and T decreases. But T cannot change too much so R has to change more, and stars move with nearly constant T.

III.8 Reactions at High Temp

When atoms can no longer exist. Need to worry about positrons or fission of nuclei.

III.8.1 Positrons

$$\gamma + \gamma \leftrightarrow e^{+} + e^{-}$$

Need $k_B T \sim m_e c^2$. Saha equation gives:

$$n(e^{-})n(e^{+}) = 4n_Q^2 e^{-2m_e c^2/k_B T}$$

Here $n(e^{-})$ is entirely determined by the background mass density, $\rho/2m_{\mathrm{H}}$. Might need to use UR n_Q, as appropriate. Example: $\rho = 10^7\,\mathrm{kg\,m^{-3}}$ and $T = 10^9\,\mathrm{K}$. $n(e^{-}) = 3 \times 10^{33}\,\mathrm{m^{-3}}$ and $n(e^{+})/n(e^{-}) \approx 1/100$. But as densities increase we cannot use this, since the limited quantum states available to the electrons (i.e., degeneracy) will inhibit formation of pairs. So this happens at high T and low(er) ρ.

One consequence of this is that it can cool the central parts of stars through neutrinos.

$$\gamma + \gamma \leftrightarrow e^{+} + e^{-} \rightarrow \nu_e + \bar{\nu}_e$$

Only rarely (10^{-22} reactions) are neutrinos produced, but they can escape right from star and take energy with them. So instead of the energy sticking around and bouncing and keeping the center of the star hot to prevent collapse the hottest regions can cool extra quickly. It doesn't actually make the star cool, but it ends up speeding up nuclear reactions (i.e., evolution).

III.8.2 Photodisintigration of Nuclei

Like ionization, but for nuclei. Ionization is at a few 1000 K, but the binding energy of a nucleus is about 10^6 times higher, so $T \sim 10^9$ K.

Example:

$$\gamma +^{20}\mathrm{Ne} \rightarrow^{16}\mathrm{O} +^{4}\mathrm{He}$$

This helium can then be important in stimulating other reactions later.

Lecture IV Reminder: Number Density

We normally think of density as mass density, ρ:

$$\rho \equiv \frac{\text{mass}}{\text{volume}}$$

But sometimes we care more about the number of things in a volume. This is number density, n:

$$n \equiv \frac{\text{number}}{\text{volume}}$$

We can relate n and ρ is we know what we are talking about. Say we have a box with hydrogen gas in it. The box has a volumne of 1 m^3. And we have 1000 atoms in the box. So the total number of atoms will be $N = 1000$. And volume $V = 1\,\mathrm{m}^3$. Which gives:

$$n = \frac{N}{V} = 1000\,\mathrm{atoms}\,\mathrm{m}^{-3}$$

Note that I wrote the units as $\mathrm{atoms}\,\mathrm{m}^{-3}$. But "atoms" aren't really a unit. They are dimensionless. So I could also have written:

$$n = \frac{N}{V} = 1000\,\mathrm{m}^{-3}$$

To get the mass density, we take:

$$\rho = \frac{M}{V}$$

But what is M? It is the total mass in the box. This is the mass of each atom times the number of them in the box. So we need to know how much mass each atom has. That is 1 amu, or 1.67×10^{-27} kg. Therefore:

$$M = 1.67 \times 10^{-27}\,\frac{\mathrm{kg}}{\mathrm{atom}} \times 1000\,\mathrm{atom} = 1.67 \times 10^{-24}\,\mathrm{kg}$$

Therefore, the mass density is:

$$\rho = 1.67 \times 10^{-24}\,\mathrm{kg}\,\mathrm{m}^{-3}$$

The quantity that helps us go between ρ and n is the average mass of the particles in our box, $\bar{m}$:

$$\rho = \bar{m}n$$

In this case the particles were all the same, so it's easy: $\bar{m} = 1.67 \times 10^{-27}\,\mathrm{kg}$. But they don't have to be the same.

In our box (still same V) we could have 1000 H atoms and 500 He atoms. We know $m_{\mathrm{H}} = 1.67 \times 10^{-27}\,\mathrm{kg}$, and $m_{\mathrm{He}} = 4 \times 1.67 \times 10^{-27}\,\mathrm{kg} = 6.69 \times 10^{-27}\,\mathrm{kg}$. The total number of "things" in the box is now $N = 1000 + 500 = 1500$, so:

$$n = 1500\,\mathrm{m}^{-3}$$

What about ρ?

$$M = 1000 \times 1.67 \times 10^{-27}\,\mathrm{kg} + 500 \times 6.69 \times 10^{-27}\,\mathrm{kg} = 5.02 \times 10^{-24}\,\mathrm{kg}$$

So we know that $\rho = 5.02 \times 10^{-24}\,\mathrm{kg\,m^{-3}}$. Which means that $\bar{m} = \rho/n = 3.34 \times 10^{-27}\,\mathrm{kg} = 2\,\mathrm{amu}$.

Or we can have the same box and have 1000 H ions, which means p and e^-. So $m_\mathrm{p} = 1.67 \times 10^{-27}\,\mathrm{kg}$, and $m_\mathrm{e} \approx 0$. The total number of "things" in the box is now $N = 1000 + 1000 = 2000$, so:

$$n = 2000\,\mathrm{m^{-3}}$$

What about ρ?

$$M = 1000 \times 1.67 \times 10^{-27}\,\mathrm{kg} + 1000 \times 0 = 1.67 \times 10^{-24}\,\mathrm{kg}$$

So we know that $\rho = 1.67 \times 10^{-24}\,\mathrm{kg\,m^{-3}}$. Which means that $\bar{m} = \rho/n = 8.36 \times 10^{-27}\,\mathrm{kg} = 0.5\,\mathrm{amu}$. So even though the total mass density is the same as the first example, the number density is different: we have twice as many "things" floating around.

You can extend this to more complicated mixtures.

Lecture V Reminder: Photons & Spectra

Spectra: disperse light through "prism", spread it out so we can see each wavelength separately.

Stars generally have *absorption line*: most of the wavelengths are bright, but a few specific wavelengths are dark. To understand this, need Kirchoff's laws:

- Hot background, cold foreground: absorption lines
- Cold background, hot foreground: emission lines

What matters is what is in front. What is in front of the star? It's is that it is hotter on the inside than the outside. So the spectrum of a star is (mostly) a blackbody with some wavelengths absorbed. These wavelengths were identified before we knew what caused them.

Fraunhofer lines: lines in Sun from things like Na, Ca. But there are also lines from H, He that are very important.

Cecilia Payne was one of the first people to identify the spectral lines in the Sun (and other stars). She showed that the elements in the Sun were very different from those on Earth: here we have almost no free H, but that is the majority of what's in the Sun.

V.2 Energy Levels for H

proton + electron in Bohr model (approaching proper quantum mechanics, but not quite): "planetary" orbits. Instead of Gravity, Coulomb force:

$$U = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

and we use the Virial theorem again, so $E = K + U = -U/2$. This would have infinite choices for r: anything is OK. The Bohr model says that r can only have particular values that are *quantized*. What is necessary is that, if you take a de Broglie wavelength, the orbit starts and stops in the same part of a wave. You can also write this as:

$$J = m_e v r = n\hbar$$

is the angular momentum. $\hbar = h/2\pi$, so this is quantum mechanical. If you do this, you get discrete energy levels for $n = 1, 2, 3, \ldots$:

$$\frac{-1}{4\pi\epsilon_0}\frac{e^2}{2r} = -\frac{1}{2}m_e v^2 = -\frac{1}{2}\frac{(n\hbar)^2}{m_e r}$$

This can only be true at certain values of r:

$$r = r_n = 4\pi\epsilon_0\frac{\hbar^2 n^2}{m_e e^2} \approx 0.5\,\text{Å}\,n^2$$

The energy levels associated with this are:

$$E_n = \frac{1}{n^2}\frac{-m_e e^4}{2(4\pi\epsilon_0\hbar)^2} = \frac{-13.6\,\mathrm{eV}}{n^2}$$

The constant $13.6\,\mathrm{eV}$ is the ionization energy of H, known as a Rydberg. Why does this ionize? Start at $n = 1$. How much energy to get to infinitely far away? This would take us to $r = \infty$, so $n = \infty$. The difference in energy levels is how much energy it would take:

$$\Delta E = E_1 - E_\infty$$

But since $E_\infty = 1/\infty = 0$, this is just E_1 or 13.6 eV.
http://astro.unl.edu/classaction/animations/light/hydrogenatom.html

V.3 Photon & Matter: Spectral Lines

Spectral lines are associated with transitions between energy levels. See in both absorption and emission.

For example, to excite an atom from $n = 1$ to $n = 2$ takes $\Delta E = hc/\lambda = E_2 - E_1 = (-3.4\,\mathrm{eV} - (-13.6\,\mathrm{eV}) = 10.2\,\mathrm{eV}$. THis gives a wavelength of $\lambda = 1216\,\text{Å}$. Lymanα. **Sketch Ly, Balmer, Pa, Brackett**. Ly$\alpha = 1216\,\text{Å}$, H$\alpha = 6563\,\text{Å}$, P$\alpha = 18,700\,\text{Å}$, Br$\alpha = 40,500\,\text{Å}$.

Emission lines: hot gas on cool background (neon light).

Absorption lines: cool gas in front of hot background.

Some astronomical objects are primarily spectral line emitters. e.g., planetary nebulae and HII regions: clouds of hot gas, where most of the emission is just what we've described.

Lecture VI Heat Transfer in Stars

Phillips Chapter 3

Heat (energy) is produced in the center, and somehow it must get to the surface. Two main ways. Either by random motion or bulk (collective) motion. Random motion can be with electrons/ions (conduction) or photons (radiative diffusion). Collective motion is rising and falling bubbles (lava lamp). This is convection and it happens when the T gradient is too steep for the other mechanisms to work.

VI.2 Heat Transfer by Random Motion

Gas with $T = T(x)$ but not too steep, so that energy flows from hot to cold slowly and everything stays close to equilibrium.

Particles move in all directions. $1/6$ move in x direction with speed v, go l before hitting something. Overall thermal energy density is $u(x)$.

Look at particles crossing x. Those coming from one side will have different energy to those from the other side if T is not the same. So energy moves.

Energy transfer per area per time is:

$$j(x) \approx \frac{1}{6}vu(x-l) - \frac{1}{6}vu(x+l) \approx -\frac{1}{3}vl\frac{du}{dx}$$

We can manipulate derivatives since $u = u(x)$ and $T = T(x)$ only, to get:

$$\frac{du}{dx} = \frac{du}{dT}\frac{dT}{dx} = C\frac{dT}{dx}$$

C is heat capacity per unit volume — depends on material. So:

$$j(x) = -K\frac{dT}{dx}$$

with $K \approx vlC/3$ is the thermal conductivity coefficient. Better calculation uses mfp $\bar{l}$ and mean speed $\bar{v}$.

VI.2.1 Electrons and Ions

For classical electrons: $u_e = (3/2)n_e k_B T$, $C_e = (3/2)n_e k_B$, and $\bar{v}_e \approx \sqrt{3k_BT/m_e}$. If the electron hits another electron not much happens — they swap energies. It is better to have electron hit an ion, so that is the mfp we need to use. This is $1/n_i\sigma_i$, where σ_i is the area for that collision. $\sigma_i \sim \pi r^2$, where r is the radius where electrostatic energy is $\sim k_BT$:

$$\frac{Ze^2}{4\pi\epsilon_0 r} \approx k_BT$$

So:

$$K_e \approx \frac{k_B}{2\pi}\frac{n_e}{n_i}\sqrt{\frac{3k_BT}{m_e}}\left(\frac{4\pi\epsilon_0 k_BT}{Ze^2}\right)^2$$

For ions can do the same, and assume $n_e = Zn_i$:

$$K_i = \frac{1}{Z^2}\sqrt{\frac{m_e}{m_i}}K_e$$

Since $Z > 1$ and $m_e \ll m_i$, $K_i \ll K_e$ and this shows that ion conduction is not important: electrons matter.

Overall this only matters in white dwarfs. There the electrons are a degenerate gas with high conductivity. Need to fix speed to be $\sim \sqrt{E_F}$ and C changes to E_F. MFP is also higher. Will do this in detail later.

VI.2.2 Photons

$u = aT^4$, $C = 4aT^3$. So $K = (4/3)c\bar{l}aT^3$. Need $\bar{l}$. At low density and high T, Thomson scattering $\bar{l} = 1/n_e\sigma_T$ with

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2$$

We can find:

$$\frac{K_\gamma}{K_e} \approx \sqrt{3}Z\frac{P_\gamma}{P_e}\left(\frac{m_ec^2}{k_BT}\right)^{5/2}$$

While P_γ is often less than the gas pressure the temperatures mean that $k_BT \ll m_ec^2$, so $K_\gamma \gg K_e$.

In more normal regimes, the absorption is more complicated. It becomes dependent on the frequency (energy) of the photon. You cannot conserve energy and momentum in a scattering between a photon and a single particle, so need at least one other. Generally interact with an electron with an ion nearby. The electron can be bound (bound-free absorption) or free (free-free absorption).

To do this properly, need to integrate to get average of $\bar{l}_\nu C_\nu$ over the photon frequencies, with $C_\nu \frac{\partial du_\nu}{\partial T}$ from the blackbody function. We end up with:

$$\bar{l} = \frac{\int d\nu\,\bar{l}_\nu C_\nu}{4aT^3}$$

which is the **Rosseland mean**. The majority of the contribution comes from where $\nu = 2.8k_BT/h$ (where C_ν is maximum) or where $\bar{l}_\nu$ is large, so that the material is transparent and a lot of energy can move.

Overall, we can write:

$$\bar{l} = \frac{1}{n_e\sigma_e + n_i\sigma_i}$$

with both densities $\propto \rho$. So we define $\bar{l} = 1/\rho\kappa$ with κ the opacity. One use is in the heat flux:

$$j(x) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{dT}{dx}$$

Calculating κ properly can be difficult.

But averaging together different mechanisms we get:

$$\kappa \propto \rho T^{-3.5}$$

which is a **Kramer's Law opacity**. At high enough temperatures we return to electron scattering, with:

$$\kappa_{es} = \frac{n_e \sigma_T}{\rho} = (1+X)\sigma_T/2m_{\mathrm{H}} \approx 0.02(1+X)\,\mathrm{m}^2\,\mathrm{kg}^{-1}$$

MFP varies from 0.07 mm to 8 mm from the center to the outside of the Sun, comparing with ~ 1 mm for the overall average.

VI.3 Convection

Bulk motion in the presence of external force (gravity). Happens on Earth too. If a blob of gas rises a bit, the material there might be cooler and denser, so buoyancy will make it continue to rise. And opposite for falling. This will transport energy efficiently, but only if the material changes properties in the right way.

Need to worry about the way T varies with x.

VI.3.1 Critical condition for convection

Ideal gas with gravity g. Background has T, P, ρ at x, and $T + \Delta T, P + \Delta P, \rho + \Delta\rho$ at $x + \Delta x$. Ideal gas law $\rho \propto P/T$ so:

$$\frac{\Delta\rho}{\rho} = \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

What about our blob? Matches at x, but might not match any more at $x + \Delta x$. Instead it will be $T + \delta T, P + \delta P, \rho + \delta\rho$. Pressure will become the same on sound-crossing timescale (fast), so $\Delta P = \delta P$. But there isn't enough time to transport heat in/out, so it changes adiabatically with $P \propto \rho^\gamma$ and

$$\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P}$$

What does buoyancy mean? That means less dense than surrounding, so $\delta\rho < \Delta\rho$. Or

$$\frac{1}{\gamma}\frac{\delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

With $\Delta P = \delta P$, convection happens if:

$$\frac{\Delta T}{T} < \frac{(\gamma-1)}{\gamma}\frac{\Delta P}{P}$$

Or, write in terms of d/dx:

$$\frac{dT}{dx} < \frac{(\gamma-1)}{\gamma}\frac{T}{P}\frac{dP}{dx}$$

Both gradients are < 0 (T and P decrease with height), so T has to fall faster than P given the adiabatic index etc.

$\gamma = 5/3$ for monatomic gas with 3 DOF. But $\gamma \to 1$ as more DOF happen. When that occurs critical gradient becomes less steep.

Also see $dP/dx = -g\rho(x)$. So if g is small then it is easier to have convection.

Convection can be quite efficient. As soon as dT/dx reaches the critical value all energy will be moved by convection and dT/dx will stay at this value. Since we have assumed adiabatic changes, the gas will have properties that are adiabatic, or isentropic.

VI.4 Temperature Gradients in Stars

Actual stars have dT/dx determined by how energy is flowing.

$$\frac{dL}{dr} = 4\pi r^2 \varepsilon(r)$$

with $\varepsilon(r)$ the energy generated per volume at r. $L(r)$ generally becomes a constant outside the core.

If radiative diffusion dominates, $L(r) = 4\pi r^2 j(r)$, so:

$$\frac{L(r)}{4\pi r^2} = -\frac{4acT(r)^3}{3\rho(r)\kappa(r)} \frac{dT}{dr}$$

The better way to think about this is that $L(r)$ is the background. If the energy is transported via radiation, then:

$$\left(\frac{dT}{dr}\right)_r = -\frac{3\rho\kappa}{4acT^3} \frac{L}{4\pi r^2}$$

For the Sun, at $0.4R_\odot$, $L = L_\odot$, $T = 5 \times 10^6\,\mathrm{K}$, $\rho = 5000\,\mathrm{kg\,m^{-3}}$, $\kappa = 0.5\,\mathrm{m^2\,kg^{-1}}$. Gives $\left(\frac{dT}{dr}\right)_r = -0.03\,\mathrm{K\,m^{-1}}$. Change in T over mfp (0.4 mm) is very small, $\Delta T/T = 2 \times 10^{-12}$, so it basically is in equilibrium.

Compare to:

$$\left(\frac{dT}{dr}\right)_c = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}$$

with $dP/dr = -g(r)\rho$. Convection dominates in ionization zones or cores of massive stars (otherwise, too much energy would need to be moved).

Ionization Zone? where ionized fraction changes relatively quickly with radius. The fact that both atoms and ions are present means that a lot of ionization/recombination happens, so energy can be absorbed by "internal DOF." Also, κ is large so a very steep T gradient would be needed for radiation to transport energy.

Sun has convection zone below surface, $R = 0.287 \pm 0.003\,R_\odot$. This is below "photosphere" where light is emitted the final time. Convective cells lead to bright/dark formations called granules in the photosphere.

Convection often present right near core, where a lot of energy is generated in a volume too small for radiation to be efficient: g is low, so dP/dr is low.

Look at $L(r)/m(r)$ (power per unit mass within r). Convection if

$$\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2} = \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{Gm(r)\rho}{r^2}$$

Use $P_r = aT^4/3$,

$$\left(\frac{L(r)}{m(r)}\right)_{\rm crit} = \frac{\gamma-1}{\gamma}\frac{16\pi Gc}{\kappa}\frac{P_r}{P}$$

If $L(r)/m(r)$ below this, radiation is enough. Otherwise need convection.

In particular this happens for massive stars where H burning is via CNO. Very temperature dependent, $\propto T^{17}$. So as $T(r)$ goes down with r, $L(r)$ goes down very rapidly. So only a small region has fusion but convection is important to move the energy out.

VI.5 WD Cooling

WD end product of non-explosive stellar evolution. Held up by electron degeneracy pressure. Roughly $R_\oplus$ and $M_\odot$, mostly C and O (that's all that fusion can do if original mass is $< 3M_\odot$ or so).

Ions are classical, electrons are quantum (degenerate). Around this is a small thin layer of "normal" matter. Cooling is via radiation from surface, but the energy gets to the surface via electron conduction. Electron conduction is very efficient, so most of the interior is effectively at a single T. Takes a long time (Gyr) to cool because k_BT of ions is high and only a little energy can leak out at a time (high opacity). Since times are Gyr, can use WDs to measure star formation and date stellar systems.

Model: hot, metal-like (fixed ions, conducting electrons) sphere surrounded by insulating jacket of ionized gas. $T = T_I$ is the same inside because degenerate electrons tranfer heat very effectively. MFP for electrons is very long (only scatter if available quantum state). Internal energy $(3/2)k_BT_I$ from ions. Jacket fixes L. Since electrons are degenerate, radius does not change as it cools since pressure is $P(n)$ only, not $P(n,T)$.

Look at outer envelope. Ideal/classical, $P = \rho k_BT/\bar{m}$. HSE and radiative diffusion give:

$$\frac{dP}{dr} = -\frac{GM\rho(r)}{r^2}$$

and

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)}{4acT(r)^3}\frac{L}{4\pi r^2}$$

L is final power at the surface since there is no energy generation. $m(r)\to M$ since we are worried about the surface layer with (almost all) of the mass inside. So combine:

$$\frac{dP}{dT} = \left(\frac{16\pi acG}{3}\frac{M}{L}\right)\frac{T^3}{\kappa}$$

What is κ? Assume Kramer's law with $X_{\rm He} = 0.9$ and the rest in heavy elements. So

$$\kappa = 4.34 \times 10^{19} \rho T^{-3.5}\,{\rm m}^2\,{\rm kg}^{-1} = \kappa_0 \rho T^{-3.5}$$

Can put in P for ρ to get $\kappa = (\kappa_0 \bar{m}/k_B) P T^{-4.5}$, so we get:

$$\frac{dP}{dT} = C\frac{T^{7.5}}{P}$$

with

$$C = \frac{16\pi a c G k_B}{3\kappa_0 \bar{m}}\frac{M}{L}$$

Integrate this through the envelope with $P = 0$ at $T = 0$:

$$\frac{P^2}{2} = C\frac{T^{8.5}}{8.5}$$

Both P and T increase going inside. When do the electrons start to become degenerate? In the plasma, $2/3$ of particles are electrons (helium). So electrons are $2/3$ of the pressure,

$$n_e = (2/3)P/k_B T = \frac{2}{3k_B}\sqrt{\frac{C}{4.25}}T^{13/4}$$

Quantum when $n_e \sim n_Q$. Highly conducting (isothermal) when $n_e = 10 n_Q$, then $T = T_I$. So that will work for the base of the envelope.

$$10\left(\frac{2\pi m_e k_B T_I}{h^2}\right)^{3/2} = \frac{2}{3k_B}\sqrt{\frac{C}{4.25}}T_I^{13/4}$$

From this,

$$T_I \approx 7\times 10^7\,{\rm K}\left(\frac{L}{M}\right)^{2/7}$$

with both L and M in solar units.

Now look at total stored energy

$$E \approx \frac{3}{2}N k_B T_I = \frac{3}{2}\left(\frac{M}{12 m_H}\right)k_B T_I$$

(assume carbon). Eventually ions will form crystal lattice, changing specific heat to $3Nk_B$ and then going lower (but $\propto T^3$). But ignoring that, $L = -dE/dt$. So

$$\frac{dT_I}{dt} = -\alpha\left(\frac{T_I}{7\times 10^7\,{\rm K}}\right)^{7/2}$$

with

$$\alpha \approx \frac{2}{3k_B}\left(\frac{12 m_H}{M_\odot}\right)L_\odot \approx 6\,{\rm K/yr}$$

Can get $T_I(t)$, and from that $L(t)$, given initial conditions (e.g., $0.4M_\odot$, $1L_\odot$, $T_I = 10^8$ K). Cooling time is $\sim$Gyr. Doing this in detail is hard (neutrinos, onset of degeneracy, solid properties, opacity, sedimentation, etc.).

Lecture VII To Make A Star

1. Support against gravity (P from HSE, E from Virial theorem)
2. Source of energy: nuclear

We can exclude all forms of energy besides nuclear fusion from powering the Sun. How does fusion work?

VII.2 Fusion

What this boils down to is $E = mc^2$: if you can get rid of a bit of mass, you liberate a lot of energy.

atomic unit $u = 1.66054 \times 10^{-27}$ kg (mass of ^{12}C/12)

proton $m_p = 1.6726 \times 10^{-27}$ kg = $1.007u = 938.8\,$MeV/c^2

neutron $m_n = 1.6749 \times 10^{-27}$ kg = $1.0087u$

electron $m_e = 9.1 \times 10^{-31}$ kg = $0.0055u$

hydrogen $m_H = 1.0078u = m_p + m_n - \mathrm{electrostaticbindingenergy}/2$

He nucleus $m_\alpha = 4.002u = 2m_p + 2m_n - \Delta m$, with $\Delta m = 0.03u \sim 0.7\% \times (4m_H) \approx 28\,\mathrm{MeV}/c^2$

So going from 4 protons to 1 He nucleus (α particle) releases 28 MeV. This is the energy released by fusion.

We can think of the binding energy as the energy released when you form something (a nucleus in this case), or as the energy that is required to break something up.

^{1}**H** : $E_b = 0$

^{4}**He** : $E_b = 28\,$MeV = $7.08\,$MeV/nucleon

^{16}**O** : $E_b = 7.97\,$MeV/nucleon

^{56}**Fe** : $E_b = 8.798\,$MeV/nucleon

^{238}**U** : $E_b = 7.3\,$MeV/nucleon

^{56}Fe has the highest binding energy, so it's the most stable. Elements that are lighter or heavier are less stable. This means that reactions would naturally squeeze lighter elements together into Fe (fusion) and break heavier elements apart (fission).

VII.2.1 Basic Nuclear Physics

1. Binding Energy: ${}^{A}_{Z}X$, with A the number of nucleons, and Z the number of protons. $E_b = (Zm_p + (A - Z)m_n - m_{\rm nuc})c^2$
2. Strong force: binds nuclei together against Coulomb (electrostatic) repulsion (since protons are positively charged)
3. $A \lesssim 56$: strong force increases faster when A increases than Coulomb forces, so a larger A leads to nuclei that are more bound.
4. $A \gtrsim 56$: the opposite

So fusion builds nuclei up to Fe, while fission breaks them down.

VII.2.1.1 H $\rightarrow$ Fe

This gives about 9 MeV/nucleon. Going from H to He gets 7 (or about 0.7% of mc^2). Going to O gets about 8 (0.8%). Going to Fe gets about 1% of mc^2 which is the most that fusion can do.

So for each proton you get $\approx 1\% mc^2 \sim 10^{-12}$ J (which means that 1 g of H could supply the annual energy of an american).

Fusion in the Sun: 10^{-12} J $\times M_\odot/m_p \sim 10^{45}$ J $\gg GM_\odot^2/R_\odot$. $t_{\rm nuc} \sim E/L_\odot \sim 10^{11}$ yr, so the Sun could shine for that long.

The actual lifespan is about 10^{10} yr (and it's lived about half of that) for a few reasons:

- $L_\odot$ increases later in life
- Not all H is burned
- It does not get hot enough to burn all the way to Fe

But it is clear that $t_{\rm nuc} \gg t_{\rm KH} \gg t_{\rm dyn}$

Lecture VIII The Nucleus

We want to explain the binding energy in $m = Zm_p + (A - Z)m_n - E_b/c^2$, where the nucleus has Z protons and $A - Z$ neutrons, for a total number A nucleons.

VIII.2 The Liquid Drop Model

$$E_B \approx a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta(A, Z)$$

Let's look at each term:

$a_V A$: this is a volume term, since for constant density nucleons the volume will be $\propto A$. This covers the binding due to the strong force, which is $\propto A$

$a_S A^{2/3}$: this is a surface term. For a volume $\propto A$, the surface area will be $\propto A^{2/3}$. It works as a correction to the volume term since the nucleons near the outside will have fewer other nucleons to interact with.

$a_C \frac{Z(Z-1)}{A^{1/3}}$: this is the Coulomb term, showing the strength of electrostatic repulsion which is $\propto 1/r \sim 1/A^{1/3}$

$a_A \frac{(A-2Z)^2}{A}$: this is an assymetry term, where nuclei with $A \approx 2Z$ are more bound

$\delta(A, Z)$: this is a pairing term

Overall this makes stable nuclei with $Z \approx A/2$, and says that the most bound nuclei are near $A = 60$. It is obviously a simplification, but it can be improved with the addition of a *shell model* (like for electrons) and using empirical data to set the various constants.

Lecture IX Fusion in Stars

Phillips Chapter 4

IX.2 The Physics of Fusion

Force two positively charged ions together. Can fuse and release excess binding energy (as photons and neutrinos). But it is very hard to force two positive charges together.

The basics are $4 \times^1 \mathrm{H} \rightarrow^4 \mathrm{He} + 28\,\mathrm{MeV}$, or releasing $\sim 7\,\mathrm{MeV}/A$. But this has some problems.

IX.2.1 Classical Result

Coulomb repulsion is strong. In order to have fusion you have to force together multiple hydrogen nuclei. These are protons, and are all positively charged. The strong force can only overcome the repulsion when the protons are *very* close together: $\sim 1\,\mathrm{fm}= 10^{-15}\,\mathrm{m}$ (for comparison, an electron orbits at $10^{-11}\,\mathrm{m}$).

The *classical* (not quantum) solution to this is that protons get close just because of their motion. They are hot, so they zip around pretty quickly. Sometimes they will approach each other, and this may happen. Can we tell how much?

The Coulomb potential is: $U_C = \frac{1}{4\pi\epsilon_0}\frac{e_1 e_2}{r}$, and energy will be conserved when the protons approach. So if they are travelling fast far away, as they approach the potential barrier they slow down:

$$\frac{1}{2}m_p v_\infty^2 + U_C(\infty) = U_C(1\,\mathrm{fm})$$

taking the limiting case that they have used all of their kinetic energy to get close enough. This gives us a requirement:

$$\frac{1}{2}m_p v_\infty^2 \geq \frac{e^2}{4\pi\epsilon_0(1\,\mathrm{fm})}$$

where we can also relate $\frac{1}{2}m_p v_\infty^2 = \frac{3}{2}k_B T$. So we need

$$T \geq T_{\mathrm{classical}} = \frac{e^2}{6k_B\pi\epsilon_0(1\,\mathrm{fm})} \sim 10^{10}\,\mathrm{K}$$

This is pretty hot, given that we know $T_c \sim 10^7\,\mathrm{K}$. So the center of the Sun is not hot enough to sustain nuclear fusion!?

In fact, Arthur Eddington proposed nuclear energy as a power source, but others thought stars were not hot enough. Eddington said: “I am aware that many critics consider the stars are not hot enough. The critics lay themselves open to an obvious retort; we tell them to go and find a hotter place.”

In the end, Eddington was right! But it needs quantum mechanics.

IX.2.2 Barrier Penetration

Two nuclei with Z_A and Z_B, m_a and m_B. At large distances interaction is Coulomb repulsion with potential

$$\frac{Z_A Z_B e^2}{4\pi\epsilon_0 r}$$

Again, only at $r < r_N = 1\,\mathrm{fm}$ does the strong force take over and make things attractive. Can write this as a Coulomb barrier: move closer until total kinetic energy is all in potential energy at $r = r_C$. This happens at energy:

$$\frac{Z_A Z_B e^2}{4\pi\epsilon_0 r_C}$$

Classically, need $r_C < r_N$:

$$E_C = \frac{Z_A Z_B e^2}{4\pi\epsilon_0 r_N} \approx \frac{1.4 Z_A Z_B}{(r_N/1\,\mathrm{fm})}\,\mathrm{MeV}$$

Compare temperature $k_B T$ to E_C. Typical central T is 10^7 K, so $k_B T \sim 1\,\mathrm{keV}$ which is $\ll 1\,\mathrm{MeV}$. There is a small tail of the Maxwell-Boltzmann distribution, but the fraction that has enough energy is $\sim e^{E_C/k_B T} = e^{-1000} = 0$. So how can fusion happen?

There is a finite chance that a nucleus can penetrate through the barrier to end up at $r < r_N$. A little bit of quantum mechanics:

$$\left(-\frac{\hbar^2 \nabla^2}{2m_r} + V(r)\right)\psi(r) = E\psi(r)$$

with $m_r = m_A m_B/(m_A + m_B)$. Then can use $\psi(r)$ to get the probability of being at different radii, $|\psi(r)|^2 4\pi r^2$.

With the Coulomb + well potential, once in the "forbidden" regime $K = E - E_C < 0$ (outside the forbidden regime it is sinusoidal). The wave function satisfies

$$\nabla^2\psi(r) = \chi^2\psi(r)$$

with

$$E = -\frac{\hbar^2\chi^2}{2m_r} + E_C$$

The solution to this is an exponentially decaying wave function, with

$$\psi(r) = \frac{e^{\chi r}}{r}$$

if there is no orbital angular momentum (radial orbit). So the probability that it gets to $< r_N$ is

$$\frac{|\psi(r_N)|^2 4\pi r_N^2}{|\psi(r_C)|^2 4\pi r_C^2} = |e^{-\chi(r_c - r_N)}|^2$$

For our purposes, the parameter χ depends on r, so we have to integrate the exponetial over $\chi(r)$. What we get is the probability is:

$$\approx e^{-\sqrt{E_G/E}}$$

with E_G the Gamow energy

$$E_G = (\pi\alpha Z_A Z_B)^2 2m_r c^2$$

and $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant.

The effect of this is that particles do not need $k_BT \sim E_C$ to undergo fusion. Some small fraction can penetrate the barrier (tunnel) and fuse. What typically happens is that $k_BT \ll E_C$, so fusion only happens for a small fraction of nuclei. The rate is effectively controlled by barrier penetration. For instance, $p+p$ has $E_G = 493\,\mathrm{keV}$, so probability is $e^{-\sqrt{E_G/k_BT}} \approx e^{-22}$ at $T = 10^7$ K. This is small, but a lot larger than e^{-1000}.

IX.2.3 Cross Sections

Just because the particles are close together doesn't mean that fusion is automatic. Need to determine cross section. i.e., if a particle goes through a medium with n particles per m^3, the probability that it hits something (reacts) when going Δx is $n\sigma\Delta x$, with σ the cross section (units of area). Probability of no reaction is $1 - n\sigma\Delta x$. So if it travels a full distance x with no reactions, the probability is the sum of all of the probabilities over Δx, $e^{-n\sigma x}$. From this, the mean free path is:

$$\bar{l} = \int_0^\infty x e^{-n\sigma x} n\sigma dx = \frac{1}{n\sigma}$$

(we used this before). In nuclear physics, typical cross section is barn, $10^{-28}\,\mathrm{m}^2$. For fusion

$$\sigma(E) = \frac{S(E)}{E} e^{-\sqrt{E_G/E}}$$

$S(E)$ is for the particular reaction and usually doesn't change much, except at resonances. To actually measure these is hard: usually we measure much closer to $k_BT = E_C$ and extrapolate down.

IX.2.4 Reaction Rates

Material with n_A and n_B, fusion cross section σ. B is fixed, but A moves with v. So time between fusion is $\tau_A = 1/n_B\sigma v$. So overall rate (density of A per time) is $R_{AB} = n_A n_B \sigma v$.

Things actually depend on relative speed v_r. This has a distribution, $P(v_r)$. What we want is:

$$\langle \sigma v_r \rangle = \int_0^\infty dv_r\, \sigma v_r P(v_r)$$

which gives things like $R_{AB} = n_A n_B \langle \sigma v \rangle$. If $A = B$ (both the same) replace $n_A n_B$ by $n^2/2$, since a proton cannot fuse with itself.

If things are thermal (Maxwell Boltzmann):

$$P(v_r) = \left(\frac{m_r}{2\pi k_B T}\right)^{3/2} e^{-m_r v_r^2/2k_B T} 4\pi v_r^2$$

And then integrate, with $E = m_r v_r^2/2$:

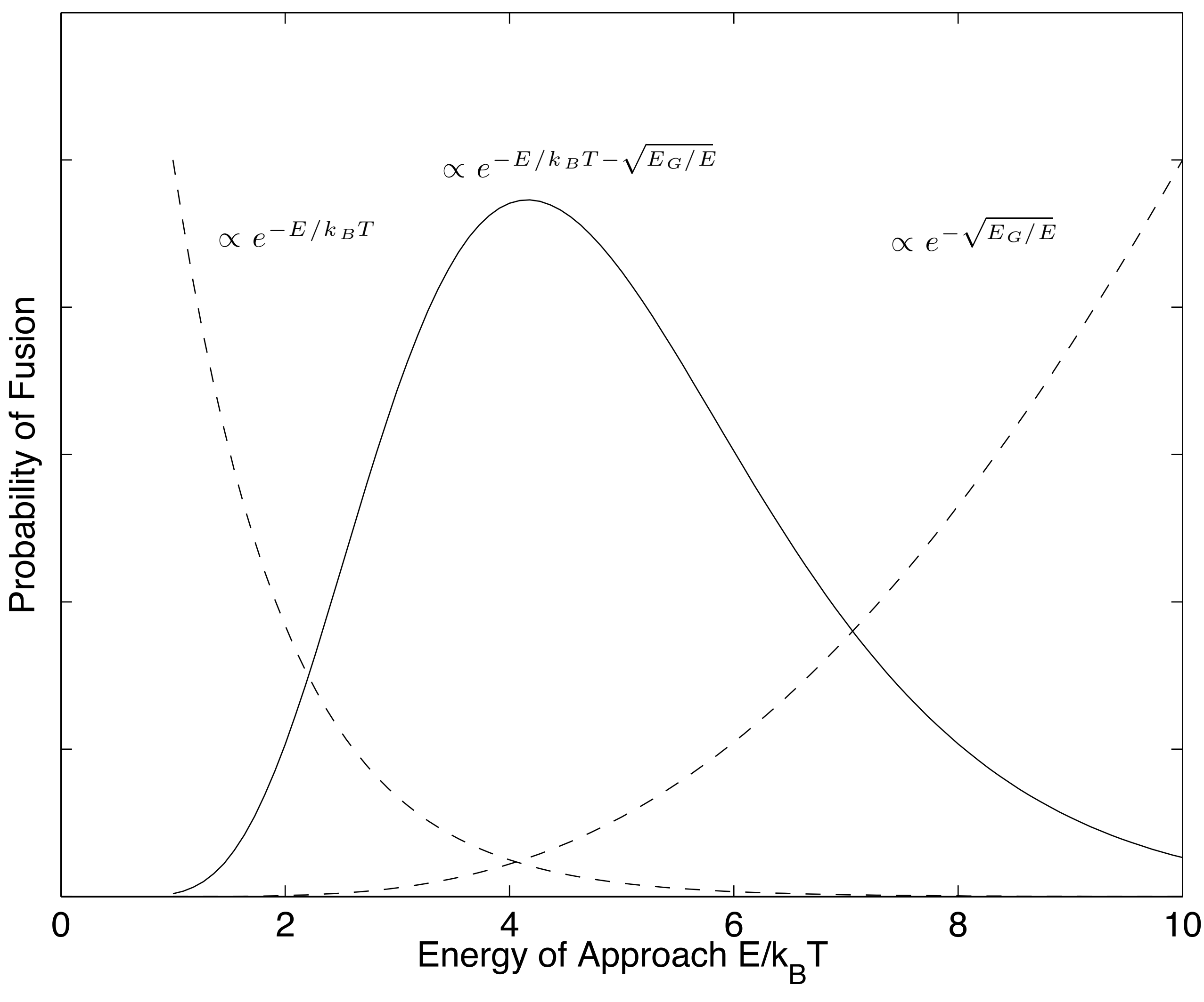
$$\langle \sigma v_r \rangle = \sqrt{\frac{8}{\pi m_r k_B^3 T^3}} \int_0^\infty dE\, E\sigma(E) e^{-E/k_B T}$$

Or,

$$R_{AB} = n_A n_B \sqrt{\frac{8}{\pi m_r k_B^3 T^3}} \int_0^\infty dE\, S(E) \exp\left(-\frac{E}{k_B T} - \sqrt{\frac{E_G}{E}}\right)$$

Look at the two factors in the integrand, ignoring $S(E)$. Product has a maximum where

$$E_0 = \left(\frac{E_G k_B^2 T^2}{4}\right)^{1/3}$$



Can determine the width of the peak by doing a Taylor expansion around it, to find:

$$\Delta = \frac{4}{3^{1/2}2^{1/3}} E_G^{1/6} (k_B T)^{5/6}$$

Where most fusion takes place around $E_0 \pm \Delta$. There are too competing factors. If the energy of approach is too low, the probability of barrier penetration is very low. If the energy is too high, the likelihood of fusion is low. Only in the middle does fusion readily occur. This is called the "Gamow Peak." For example, at 2×10^7 K and for fusing two protons, $E_0 = 7.2\,\mathrm{keV}$ or $4.2\,k_B T$, and $\Delta/2 = 4.1\,\mathrm{keV}$.

If we assume $S(E)$ is constant we can evaluate it at E_0 and do the integral for R_{AB}:

$$R_{AB} \approx 6.48 \times 10^{-24} \frac{n_A n_B}{A_r Z_A Z_B} S(E_0) \left(\frac{E_G}{4 k_B T} \right)^{2/3} e^{-3(E_G/4k_BT)^{1/3}} \,\mathrm{m}^{-3}\,\mathrm{s}^{-1}$$

where $A_r = m_r/\mathrm{amu}$ and the units of $S(E_0)$ are keV barns.

The strongest effect of changing T comes from the exponential term. As T increases, R_{AB} will go up a lot. We can take an approximate derivative to see how quickly it increases:

$$\frac{dR_{AB}}{dT} \approx \left(\frac{E_G}{4 k_B T} \right)^{1/3} \frac{R_{AB}}{T}$$

which is for proton + deuteron (an important step in making He) $\frac{dR_{AB}}{dT} \approx 4 \frac{R_{AB}}{T}$. Which implies $R_{AB} \propto T^4$. This is steep, but other reactions can be even steeper. For instance, proton + carbon is $\propto T^{17}$.

Lecture X Hydrogen Burning

What are the actual reactions that take place? First fusion is around 10^6 K, with light elements such as D and Li. Those are fast and easy, but there are not much of them. Need to turn H into He.

The problem is that to go from 4 protons to a ^{4}He, we need to turn protons to neutrons. This can generally proceed via interactions involving the weak nuclear force, such as $p \to n + e^+ + \nu_e$. And weak reactions are (relatively) slow (if see neutrinos, then weak). So how does it work?

X.2 Proton Proton Chain

Worked out by Bethe in 1939. First reaction is

$$p + p \to d + e^+ + \nu_e$$

which is inverse beta decay where the neutron is immediately captured. After this it can take a number of routes.

$$p + p \to d + e^+ + \nu_e$$
$$p + d \to^3 \mathrm{He} + \gamma$$

Branch I	Branch II	Branch III
$^3\mathrm{He} +^3 \mathrm{He} \to^4 \mathrm{He} + 2p$	$^3\mathrm{He} +^4 \mathrm{He} \to^7 \mathrm{Be} + \gamma$	
	$e^- +^7 \mathrm{Be} \to^6 \mathrm{Li} + \nu_e$	$p +^7 \mathrm{Be} \to^8 \mathrm{B} + \gamma$
	$p +^7 \mathrm{Li} \to^4 \mathrm{He} +^4 \mathrm{He}$	$^8\mathrm{B} \to^8 \mathrm{B}^* e^+ + \nu_e$
		$^8\mathrm{Be}^* \to^4 \mathrm{He} +^4 \mathrm{He}$
Branch I	Branch II	Branch III
$Q = 26.2\,\mathrm{MeV}$	$Q = 25.2\,\mathrm{MeV}$	$Q = 19.1\,\mathrm{MeV}$
85%	15%	0.02%

The net energy Q accounts for the mass converted into energy and for the energy released by annihilating positrons, but not the energy carried away by neutrinos.

The first step is slow (weak force). Then that d is immediately converted into $^3\mathrm{He}$ via a reaction whose S factor is 18 orders of magnitude greater (strong force). Original protons take a long time to react (9×10^9 yr), but d will react in 1 s.

This has a consequence for the deuterium we find on Earth. The equilibrium d in the Sun is tiny, roughly 10^{-18} of the p concentration (set by the relative rates). On Earth it is closer to 0.015%. So it cannot come from stars, as has to be primordial (from big bang).

^{4}He in branches II and III are catalyst: one is returned for every one that is used.

Rate is determined by first reaction. Can combine to get:

$$\epsilon_{pp} = 9.5 \times 10^{-37} X_{\mathrm{H}}^2 \rho^2 T^4 \,\mathrm{W\,m^{-3}}$$

energy per time per volume.

X.3 CNO Cycle

For stars like the Sun, pp is enough. But as stars go up in mass the luminosity goes up much too quickly for pp to be doing the work: the central temperatures do not go up very much, but L does. So what is happening? We need a reaction chain that has a stronger T dependence, and therefore has a higher Coulomb barrier. This would also explain why it doesn't operature in lower-mass stars: they just don't have enough energy to get over the barrier.

If the barrier is higher, that will come from more protons, which means heavier elements. It cannot consume those elements since there are not many of them, but it must recycle them.

Makes use of carbon from earlier He burning (previous generation of stars). Uses CNO in a cycle, but net result is H to He.

$$\begin{aligned}
p +^{12}\mathrm{C} &\rightarrow {}^{13}\mathrm{N} + \gamma \\
{}^{13}\mathrm{N} &\rightarrow {}^{13}\mathrm{C} + e^{+} + \nu_e \\
p +^{13}\mathrm{C} &\rightarrow {}^{14}\mathrm{N} + \gamma \\
p +^{14}\mathrm{N} &\rightarrow {}^{15}\mathrm{O} + \gamma \\
{}^{15}\mathrm{O} &\rightarrow {}^{15}\mathrm{N} + e^{+} + \nu_e \\
p +^{15}\mathrm{N} &\rightarrow {}^{12}\mathrm{C} +^{4}\mathrm{He}
\end{aligned}$$

Sequence of proton captures and inverse beta decays. Net result is $4p \rightarrow^{4}\mathrm{He} + 2e^{+} + 2\nu_e + Q$, with $Q = 23.8\,\mathrm{MeV}$. This is the dominant cycle, but others are possible.

The slowest reaction (which determines the rate) is $p +^{14}\mathrm{N} \rightarrow^{15}\mathrm{O} + \gamma$. The mean lifetime of ^{14}N is 5×10^{8} yr in the Sun.

But overall the fusion rate is $\propto T^{18}$, so for higher M (higher T) the rate is faster.

Once a star explodes, some of the ^{12}C that had been made through He burning will be stuck as ^{13}C, ^{14}N, and ^{15}N, especially ^{14}N.

X.4 Solar Neutrinos

How do we know? We measure M, R, L, and T_{eff}. We can get at some properties though comparison with other stars. But can we be certain? Neutrinos tell us what is inside.

Net reaction for Sun:

$$4p \rightarrow^{4}\mathrm{He} + 2e^{+} + 2\nu_e$$

So every He has 2 neutrinos, and we can relate this to the luminosity through the Q factor: $2L_{\odot}/Q = 1.86 \times 10^{38}\,\mathrm{s}^{-1}$.

During escape from the Sun, probability of interaction is $n\sigma R_{\odot}$, with $n \sim 10^{30}\,\mathrm{m}^{-3}$ and $\sigma \sim 10^{-48}\,\mathrm{m}^{2}$ the cross section for interaction. So find only $\sim 10^{-9}$ will interact. Expect $F_{\nu} = 6.6 \times 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$.

But these neutrinos carry more information, since different reactions produce neutrinos with different energies. In particular, the majority of the neutrinos come out with low energy, 0.4 MeV, which is very hard to detect. Easier to detect more energetic neutrinos ($> 10\,\mathrm{MeV}$) from ${}^8\mathrm{B} \rightarrow {}^8\mathrm{Be} + e^+ + \nu_e$.

It is hard to detect neutrinos. If the Sun cannot stop them, how can we? Employ reactions like

$$\nu_e + {}^{37}\mathrm{Cl} \rightarrow {}^{37}\mathrm{Ar} + e^+$$

with a whole bunch of Cl coming from cleaning fluid. For this, $\sigma = 10^{-46}\,\mathrm{m}^{-2}$, so rate is 6×10^{-36} per second per Cl atom. Define 1 SNU is as 1 capture per 10^{36} atoms per second, so expect 6.1 SNU from main reaction. Overall, expect 7.9 ± 2.6 SNU. But see $2.55 \pm 0.17 \pm 0.18$ SNU (based on 1 interaction per day).

This is only $1/3$ what is predicted. Is the Sun broken? Multiple experiments have confirmed this deficiency. Look at other neutrinos energies, other reactions. Same result.

Look at reactions with other kinds of neutrinos. In particular, charged current:

$$\nu_e + {}^2\mathrm{H} \rightarrow p + p + e^-$$

(electron neutrinos only) and neutral current

$$\nu_x + {}^2\mathrm{H} \rightarrow n + p + \nu_x$$

(all types). Needs a large amount of ^{2}H, done in Sudbury Neutrino Observatory with 1000 tons of heavy water. Find results that agree with standard solar model, confirmed with other experiments at the same time that imply the previous discrepancy is still there. [Kamiokande can record time/direction of neutrinos, sees that they come from the Sun.]

What is happening is neutrino oscillations: if neutrinos have (a little) mass, there is a chance they will oscillate from one type (electron) to another (muon, tau). Because mass differences are small, length for oscillations is large. Has important implications for cosmology, supernovae, etc. Since confirmed by other experiments.

X.5 Helium Burning

Once H burning has ceased in a part of the star, it is mostly just He sitting around waiting for things to get hot/dense enough for fusion to happen. This fusion is very important, since it makes C and O (18% and 65% of your body; 0.39% and 0.85% of the solar system).

After H burning, the center of the star is He. It contracts, releases gravitational energy into KE. Half escapes, half into heat. Eventually H burning starts again but in a shell around the core, making more He.

He burning can start if $M > 0.5\,M_\odot$ or so, when core is $> 10^8$ K and $\rho = 10^{5-8}\,\mathrm{kg\,m^{-3}}$.

This dramatically changes the appearance of the star. Increase in T leads to a large increase in the outer layers. When He burning starts the core expands and cools, shrinking the outer envelope a bit. Net effect is a dense, burning core and a large extended envelope: a red giant.

He burning would be easy if there were stable nuclei with mass of 5 or 6 amu. Then would just do proton capture onto He. But something must happen. Discussed by Salpeter in 1952.

$$^4\mathrm{He} +^4 \mathrm{He} \leftrightarrow^8 \mathrm{Be}$$

makes unstable Be nuclei in equilibrium. Some of this then:

$$^4\mathrm{He} +^8 \mathrm{Be} \leftrightarrow^{12} \mathrm{C}^*$$

make unstable exciting C. Finally,

$$^{12}\mathrm{C}^* \rightarrow^{12} \mathrm{C} + \begin{cases} 2\gamma \\ e^+ + e^- \end{cases}$$

where the exciting C decays into stable C. The net reaction is then:

$$^4\mathrm{He} +^4 \mathrm{He} +^4 \mathrm{He} \rightarrow^{12} \mathrm{C}$$

with a net energy release of $7.275\,\mathrm{MeV}$. This is called the triple-alpha process. The first two steps happen in equilibrium, creating and destroying unstable nuclei. To derive the rate we can look at each step separately.

X.5.1 Production of Be

Be is slightly unstable, since $(m_8 - 2m_4)c^2 = 91.8\,\mathrm{keV} > 0$ (the Be is more massive than two He). The decay happens with a lifetime of $\tau = 2.6 \times 10^{-16}$ s. But $91.8\,\mathrm{keV}$ is a relatively small amount of energy compared to the MeV released by fusion.

Moreover, this amount of energy is within the $E_0 \pm \Delta/2$ window for favorable fusion. $E_G = 31.6\,\mathrm{MeV}$, so for 1×10^8 K the window is $81 \pm 31\,\mathrm{keV}$, and for 2×10^8 K it is $132 \pm 55\,\mathrm{keV}$. The Be would tend to decay right away, but if there are enough of them around and the reaction reaches steady state we can look at the Saha equilibrium:

$$\mu_A = m_A c^2 - k_B T \ln\left(\frac{g_A n_Q}{n_A}\right)$$

with

$$n_Q = \left(\frac{2\pi m_A k_B T}{h^2}\right)^{3/2}$$

giving

$$\frac{n_8}{n_4^2} = 2^{3/2}\left(\frac{h^2}{2\pi m_4 k_B T}\right)^{3/2} e^{-\Delta mc^2/k_B T}$$

So in a plasma with 2×10^8 K, $\rho = 10^8\,\mathrm{kg\,m^{-3}}$, $n_4 = \rho/m_4 = 1.5 \times 10^{34}\,\mathrm{m^{-3}}$ (almost all He) but $n_8 = 7 \times 10^{26}\,\mathrm{m^{-3}}$, or $1/2 \times 10^7$ of the He. The ratio goes down a lot with temperature, reaching $1/2 \times 10^9$ at 1×10^8 K. There is not much Be, but it is enough.

X.5.2 Production of Excited C

Make $^{12}\mathrm{C}^*$ in 0^+ excited state. Was only hypothesized to explain He burning (by Fred Hoyle). Need an extra bit of fusion to make sure that stars with 1.2×10^8 K can burn, and that the resonance should be at about 300 keV above fusion threshold (i.e., the $S(E)$ factor is not smooth). Resonance is at 7.65 MeV, so

$$(m_{12}^* - m_{12})c^2 = 7.6542\,\mathrm{MeV}$$

This is very close to the energy for He + Be, and also close to the energy for 3 He:

$$(m_{12}^* - m_4 - m_8)c^2 = 287.7\,\mathrm{keV}$$

$$(m_{12}^* - 3m_4)c^2 = 379.5\,\mathrm{keV}$$

The 287.7 keV difference is close to the fusion window for gas at $> 10^8$ K. So Saha equation again:

$$\frac{n_{12}^*}{n_4 n_8} = \left(\frac{3}{2}\right)^{3/2} \left(\frac{h^2}{2\pi m_4 k_B T}\right)^{3/2} e^{-\Delta m c^2/k_B T}$$

and can relate n_8 to n_4 from before:

$$\frac{n_{12}^*}{n_4^3} = 3^{3/2} \left(\frac{h^2}{2\pi m_4 k_B T}\right)^{3} e^{-(m_{12}^* - 3m_4)c^2/k_B T}$$

So the He has to go through Be, but it doesn't stay there, since the rate of production and destruction depend on the other channels. We could have looked at

$$3 \times ^4\mathrm{He} \leftrightarrow ^{12}\mathrm{C}^*$$

directly.

The concentration of excited C is low, about $3 \times 10^{14}\,\mathrm{m}^{-3}$ at 2×10^8 K

X.5.3 Carbon Production

Most of the reactions happen back the way they came:

$$3 \times ^4\mathrm{He} \leftrightarrow ^4\mathrm{He} + ^8\mathrm{Be} \leftrightarrow ^{12}\mathrm{C}^*$$

But some of the excited C can decay to normal C by emitting two gamma-rays or electron/positron pair, with a time of 1.8×10^{-16} s, releasing 7.65 MeV. This doesn't really change the equilibrium concentrate since only $\approx 1/2500$ do this. We can look at the rate of carbon production:

$$\frac{dn_{12}}{dt} = \frac{n_4^3}{\tau} 3^{3/2} \left(\frac{h^2}{2\pi m_4 k_B T}\right)^{3} e^{-(m_{12}^* - 3m_4)c^2/k_B T}$$

The physics in here is the decay timescale and the mass difference, both of which have been measured.

So

$$\epsilon_{3\alpha} = (3m_4 - m_{12})c^2 \frac{dn_{12}}{dt}$$

As an example, 2×10^8 K and $10^8\,\mathrm{kg\,m^{-3}}$. So $n^*_{12} = 3 \times 10^{14}\,\mathrm{m^{-3}}$. $dn_{12}/dt = 1.9 \times 10^{30}\,\mathrm{m^{-3}\,s^{-1}}$, and $\epsilon_{3\alpha} = 2.2 \times 10^{18}\,\mathrm{W\,m^{-3}}$. But this is very temperature sensitive, since the activation energy $379.5\,keV \gg k_B T$ (10 keV). Overall $\epsilon_{3\alpha} \propto T^{41}$.

X.5.4 What Happens to the Carbon

Carbon burning:

$$^4\mathrm{He} +^{12}\mathrm{C} \rightarrow^{16}\mathrm{O} + \gamma$$

(which is good for us). Straightforward reaction w/o resonance or unstable meta-state. Then

$$^4\mathrm{He} +^{16}\mathrm{O} \rightarrow^{20}\mathrm{Ne} + \gamma$$

Can have additional He captures to make other elements (Ne, Mg, Si), although this doesn't happen too much during He burning since the temperature isn't high enough.

These reactions bypass Li, Be, B. We do not see very much of these, and most of what we do see comes not from stars but from a cosmic ray hitting a heavy nucleus and splitting it (spallation).

X.5.5 Advanced Burning

Hotter and hotter. Core of C and O builds up (if the star isn't massive enough, this ends up as WD). C burning starts at 5×10^8 K, with reactions like:

$$^{12}\mathrm{C} +^{12}\mathrm{C} \rightarrow^{20}\mathrm{Ne} +^4\mathrm{He}$$

(or $^{23}\mathrm{Na}$ or $^{23}\mathrm{Mg}$). If the star is a little more massive (8–10 $M_\odot$) things can ened here with O/Ne/Mg WD.

Then Ne burning if $> 10^9$ K, making $^{24}\mathrm{Mg}$. Important next step is

$$^{16}\mathrm{O} +^{16}\mathrm{O} \rightarrow^{28}\mathrm{Si} +^4\mathrm{He}$$

($> 2 \times 10^9$ K) followed by Si burning (3×10^9 K). The reactions mostly involve heavy nuclei + light particles made from breaking up the heavier ones, gets rather complicated.

This break-up happens when photons have enough energy to split apart nuclei. E.g.,

$$\gamma +^{28}\mathrm{Si} \rightarrow^{24}\mathrm{Mg} +^4\mathrm{He}$$

Just like ionization, but at temperatures 10^6 times higher.

All this happens very quickly. And it needs a very massive star to keep going, where everything happens more quickly than in the Sun. For instance, for a $25\,M_\odot$ star:

Stage	Timescale	$T/10^9$ K	ρ
H burning	7×10^6 yr	0.06	5×10^4
He burning	5×10^5 yr	0.23	7×10^5
C burning	600 yr	0.93	2×10^8
Ne burning	1 yr	1.7	4×10^9
O burning	6 mo	2.3	1×10^{10}
Si burning	1 day	4.1	3×10^{10}

Then things get exciting!

Lecture XI Stellar Structure

Phillips 5

Put together the different pieces we've assembled. For instance, HSE:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

with mass defined by:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

Energy transport (radiative):

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T(r)^3}\frac{L(r)}{4\pi r^2}$$

and energy from:

$$\frac{dL}{dr} = 4\pi r^2 \epsilon(r)$$

These are what we need to make a star. Ignores convection, spherical, equilibrium. But we also need to know some other bits:

$$P = P(\rho, T)$$
$$\kappa = \kappa(\rho, T)$$
$$\epsilon = \epsilon(\rho, T)$$

Look at pressure from classical particles:

$$n_e = (1 + X_{\rm H})\frac{\rho}{2m_{\rm H}}$$

$$n_i = (2X_{\rm H} + 0.5X_{\rm He})\frac{\rho}{2m_{\rm H}}$$

along with

$$n = n_e + n_i = (1 + 3X_{\rm H} + 0.5X_{\rm He})\frac{\rho}{2m_{\rm H}}$$

so

$$P = n_e k_B T + n_i k_B T = n k_B T$$

This can become degenerate

$$P = K_{\rm NR} n_e^{5/3} \rightarrow K_{\rm UR} n_e^{4/3}$$

And radiation pressure:

$$P = \frac{a}{3}T^4$$

Opacity from electron scattering (higher temp, lower density):

$$\kappa_{\rm es} = 0.02(1 + X_{\rm H})\,{\rm m}^2\,{\rm kg}^{-1}$$

or bound-free/free-free:

$$\kappa \propto \rho T^{-3.5}$$

And energy:

$$\epsilon_{pp} = 9.5 \times 10^{-37} X_{\rm H}^2 \rho^2 T^4\,{\rm W\,m}^{-3}$$

XI.1.6 Vogt Russell Theorem

The mass and composition (as a function of r) of a star uniquely determine the radius, luminosity, structure; and then the evolution.

We can say this because the properties of a given layer in a star allow us to integrate the equation inward, and must match boundary conditions. So for a given set of ingredients, there is only a single star that can be made.

XI.2 Simple Stellar Models

Want to put all of these together. Make into 4 coupled first-order ODEs $P(r)$, $m(r)$, $T(r)$, $L(r)$. Need boundary conditions. Some are easy: $m(0) = L(0) = 0$ (no mass inside that). At the outside, $P(R)$ and $T(R)$ need to merge into the photosphere which is complicated. We will ignore that for the moment, assume $T(R) = P(R) = 0$.

Combine HSE and mass into:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Second order in $P(r)$, $\rho(r)$. Assume a simple relation between these:

$$P = K\rho^\gamma = K\rho^{(n+1)/n}$$

This is a **polytrope** with index n, with $\gamma = (n+1)/n$. So we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\left(K\rho^{(n+1)/n}\right)\right) = -4\pi G\rho$$

We can now make our additional boundary conditions $\rho(0) = \rho_c$, $d\rho/dr(0) = 0$. This sets the central density, and says that there is not a cusp of material. The outer boundary comes from having ρ go to 0, or $\rho(R) = 0$ and $m(R) = M$.

These models are overly simple, but can still be useful. Especially before computers. Let us work a bit on the math.

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left(r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Let us simplify the units. $\rho(r) = \rho_c(D_n(r))^n$, where $D_n(r)$ is a function that goes between 0 and 1. So:

$$\left((n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$$

The bit out in front has units of distance squared. So:

$$\lambda_n \equiv \left((n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right)^{1/2}$$

and normalize:

$$\xi \equiv \frac{r}{\lambda_n}$$

So we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dD_n}{d\xi}\right) = -D_n^n$$

This is the **Lane-Emden** equation for a polytrope. We have written it in terms of dimensionless variables $D_n(\xi)$ to make a physics problem into a math problem, but we must be careful to put the units back in before we give physics results.

Boundary conditions as before, but also stop the integration where $D_n(\xi) = 0$. This is the first 0 of the function, and defines the outer edge at $\xi = \xi_1$.

To compute the mass:

$$M = 4\pi\int_0^R dr\,\rho r^2 = 4\pi\int_0^{\xi_1} d(\lambda_n\xi)\,(\lambda_n\xi)^2\rho_c D_n^n = 4\pi\lambda_n^3\rho_c\int_0^{\xi_1} d\xi\,\xi^2 D_n^n$$

We don't necessarily have to solve for D_n and integrate to get this, since we can recognize that $\xi^2 D_n^n = -d/d\xi(\xi^2 dD_n/d\xi)$, so

$$M = -4\pi\lambda_n^3\rho_c\xi_1^2\frac{dD_n}{d\xi}|_{\xi_1}$$

Numerically this is useful, but there are a few analytic solutions. Namely, $n = 0$, 1, and 5. For $n = 1$ the solution is:

$$D_1(\xi) = \frac{\sin\xi}{\xi}$$

where we only do it up to the first zero, $\xi_1 = \pi$. And for $n = 5$ there is no finite radius:

$$D_5(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

with $\xi_1 = \infty$. However, the total mass is finite. For $n > 5$ the mass is infinite.

For adiabatic monatomic gas, $\gamma = 5/3$ and $n = 1.5$. This also works for white dwarfs in some cases.

$n = 3$ is useful, since this is what happens for a star in radiative equilibrium. Add radiative and gas pressure, $P_g = \rho k_B T/\bar{m} = \beta P$, $P_r = aT^4/3 = (1-\beta)P$. Eliminate T in favor of β:

$$\frac{a}{3}\left(\frac{\beta P\bar{m}}{\rho k_B}\right)^4 = (1-\beta)P$$

So from here you can see how $P = K\rho^{4/3}$ comes out.

XI.2.1 Another Way

Look at the pressure distribution. At the center of the Sun, it is 2×10^{16} Pa, which is about 200 times the average.

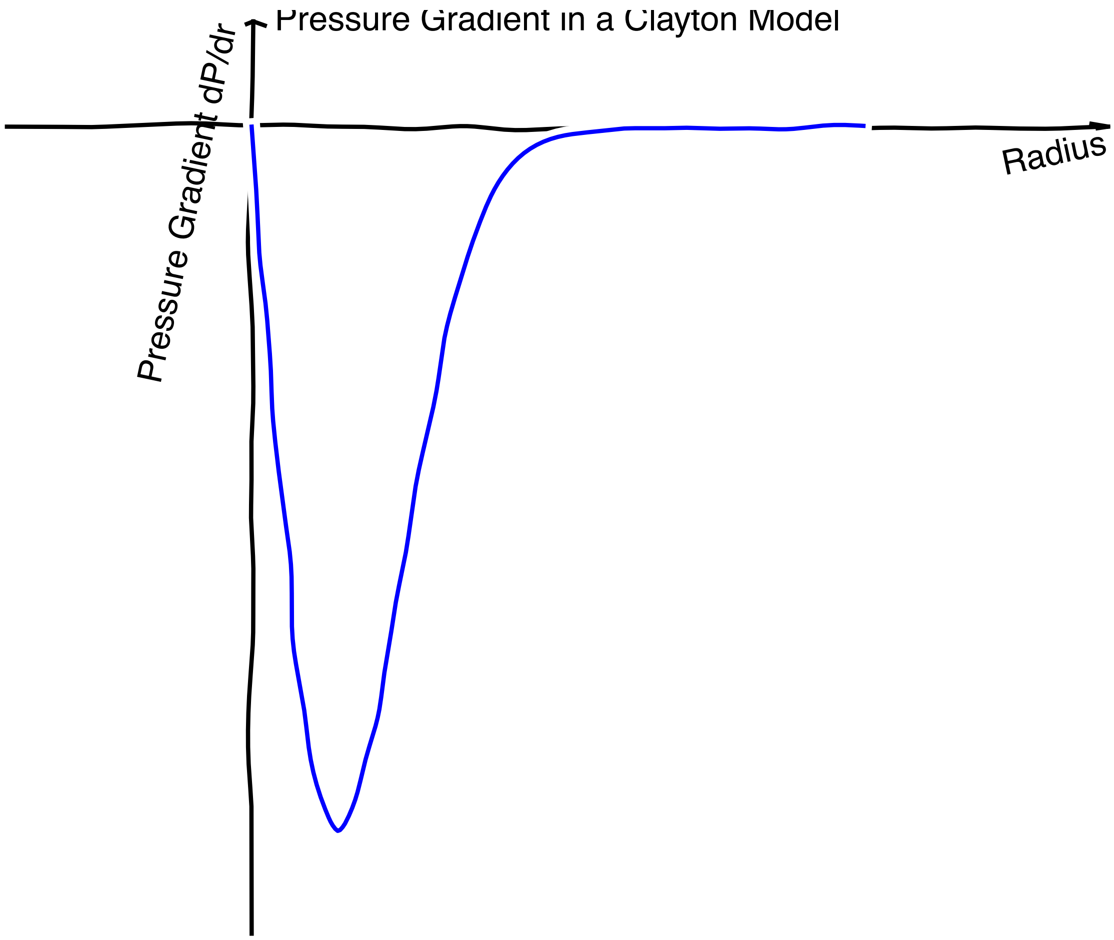
Can write pressure gradient:

$$\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_c^2 r$$

near the center with constant density ρ_c from HSE. Near the outside the total mass becomes constant M:

$$\frac{dP}{dr} = -\frac{GM\rho(r)}{r^2}$$

Somehow these have to join.


Pressure Gradient in a Clayton Model
Pressure Gradient dP/dr
Radius

Guess a form:

$$\frac{dP}{dr} = -\frac{4\pi}{3} G\rho_c^2 r e^{-r^2/a^2}$$

This will roughly conform to our limits above. Not great at large r, but won't be horrible. Minimum of gradient is at $r = a/\sqrt{2}$.

Integrate to get:

$$P(r) = \frac{2\pi}{3} G\rho_c^2 a^2 \left(e^{-r^2/a^2} - e^{-R^2/a^2}\right)$$

From here, can get ρ, T. Use HSE and mass equation to get:

$$Gm(r)dm = -4\pi r^4 dP$$

or

$$G\frac{m^2(r)}{2} = -4\pi \int_0^r dr' r'^4 \frac{dP}{dr'}$$

and then use the gradient we derived to get:

$$m(r) = \frac{4\pi a^3}{3} \rho_c \Phi\left(\frac{r}{a}\right)$$

with

$$\Phi^2(x) = 6\int_0^x dx' x'^5 e^{-x'^2} = 6 - 3(x^4 + 2x^2 + 2)e^{-x^2}$$

Then:

$$\rho(r) = \frac{1}{4\pi r^2}\frac{dm}{dr} = \rho_c \left(\frac{x^3 e^{-x^2}}{\Phi(x)}\right)$$

with $x = r/a$. To get T need EOS. But if ideal gas .... In particular, can expand $\Phi(x)$ around $x = 0$ to get series expressions for ρ, T near the center.

XI.2.1.1 A Star with a High Central Density

If $a \ll R$, ignore e^{-a^2/R^2} terms. This when the mass is all near the center. Works OK for Sun, with $a = R_\odot/5.4$.

Then

$$M \approx \frac{4\pi\rho_c a^3\sqrt{6}}{3}$$

Can determine $\bar{\rho}$, $\rho(a)$, $m(a)$, etc. At the center,

$$P_c \approx \frac{2\pi}{3} G\rho_c^2 a^2 \approx \left(\frac{\pi}{36}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$$

So with the pressure at the center we know the density, and the other way. Should work for any star that is homogeneous and concentrated.

We can get similar expressions valid for other sorts of models. For instance, polytrope with $n = 1.5$ ($\gamma = 5/3$):

$$P_c = 0.48 G M^{2/3} \rho_c^{4/3}$$

$n = 3$:

$$P_c = 0.36 G M^{2/3} \rho_c^{4/3}$$

Overall, there is an upper bound:

$$P_c < \left(\frac{\pi}{6}\right)^{1/3} G M^{2/3} \rho_c^{4/3}$$

And if we have these relations, we can also try to determine T_c (and hence fusion, etc.).

XI.3 Modeling the Sun

Detailed modeling: $P_c = 1.65 \times 10^{16}$ Pa, $\rho_c = 9.0 \times 10^{14}\,\mathrm{kg\,m^{-3}}$, $T_c = 13.7 \times 10^6$ K (Strömgren).

With our simple model, need ρ_c, a, $R = R_\odot$. Also set $M = M_\odot$ and ρ_c from Strömgren. From there get $a = R_\odot/5.4$. Gives $P_c = 1.9 \times 10^{16}$ Pa, a little high. Get T_c from ideal gas law, 16×10^6 K.

XI.3.1 Luminosity

Can integrate ϵ_{pp} and also look at L carried by radiation. This is done for early "homogeneous" Sun: now things are different since burning has been going on for a while.

$$L_\odot \approx 8.4 \times 10^{-37}\,\mathrm{W} X_{\mathrm{H}}^2 \int_0^{R_\odot} dr\, 4\pi r^2 \rho(r)^2 T(r)^4 \approx 1.0 \times 10^{-36}\,\mathrm{W} a^3 X_{\mathrm{H}}^2 \rho_c^2 T_c^4$$

Can substitute in, find $L_\odot \approx 5 \times 10^{26}$ W, compared to predicted 3×10^{26} W (was a little different then).

Compare to heat flow. Assume Kramer's law opacity, $\kappa(r) \approx \kappa_c(1 + 11r^2/16a^2)$. Then:

$$L(r) \approx 16\pi\sigma \frac{T_c^4 r^3}{\kappa_c \rho_c a^2}\left(1 - \frac{19r^2}{16a^2}\right) \approx 3 \times 10^{29}\,\mathrm{W} \frac{r^3}{R_\odot^3}\left(1 - 35\frac{r^2}{R_\odot^2}\right)$$

Obviously this fails for r comparable to $R_\odot$. But our radiative equilibrium solution is really only valid in this model for r near the core (we did not enforce equilibrium on $T(r)$). So $L(R_\odot/10) \approx 2 \times 10^{26}$ W, which is close.

XI.4 Minimum and Maximum Masses

Most of the stars we know are from $0.1\,M_\odot$ to $50\,M_\odot$. Why? Is there anything fundamental that gives us the scale of $M_\odot$, and what causes these limits?

Require HSE. From that, we get:

$$P_c \approx \left(\frac{\pi}{6}\right)^{1/3} G M^{2/3} \rho_c^{4/3}$$

All this requires is that the star is the same chemically throughout (not precisely true) and that the mass increases toward the center.

XI.4.1 Minimum Mass

Need central conditions extreme enough to sustain pp burning. Consider a collapsing cloud of mass M. Kelvin-Helmholtz contraction, so all of energy is from contraction (gravity) not fusion. Looks like an ideal gas:

$$P_c = \frac{\rho_c}{\bar{m}} k_B T_c$$

The contraction will be slow and close to HSE if the pressure is almost enough to balance the star. Equating the two pressures:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G\bar{m}M^{2/3}\rho_c^{1/3}$$

So $T_c \propto \rho_c^{1/3}$, which goes up during contraction. Contraction will continue until T is enough for fusion or electrons become degenerate — either way the center will be supported against further contraction. So it will not be a star if center is degenerate before fusion.

Assume that electrons have become degenerate. Then:

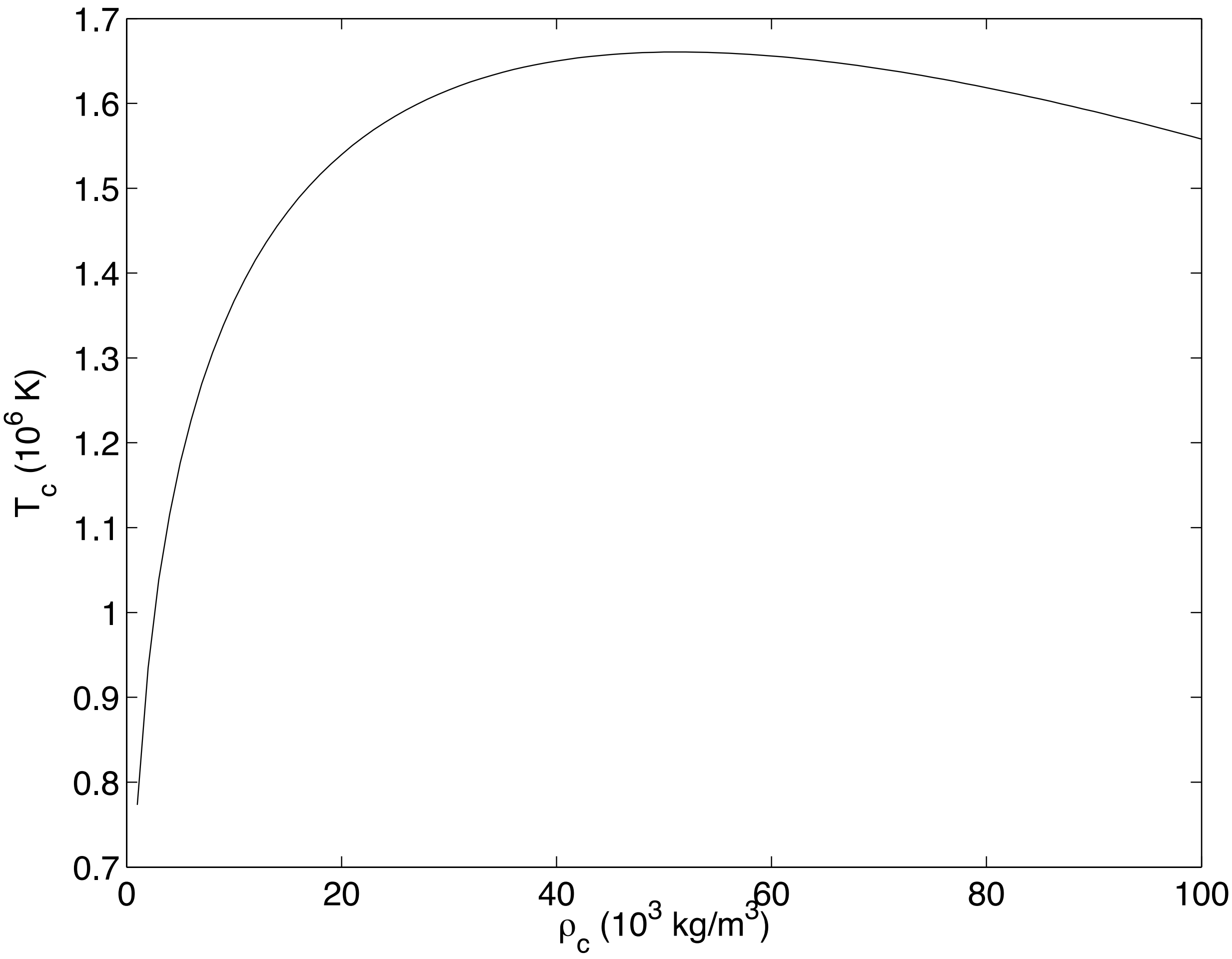
$$P_c = K_{\rm NR} n_e^{5/3} + n_i k_B T_c \approx K_{\rm NR}\left(\frac{\rho_c}{m_{\rm H}}\right)^{5/3} + \frac{\rho_c}{m_{\rm H}} k_B T_c$$

Set this equal to our P_c from before:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G m_{\rm H} M^{2/3}\rho_c^{1/3} - K_{\rm NR}\left(\frac{\rho_c}{m_{\rm H}}\right)^{2/3}$$

So this is the temperature when the electrons are degenerate but the ions are not. What is the maximum temperature that will be reached?

$$k_B T_c = A\rho_c^{1/3} - B\rho_c^{2/3}$$



1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
T_c (10^6 K)
0
20
40
60
80
100
ρ_c (10^3 kg/m^3)

Can differentiate and find maximum. This is at $k_B T_c = A^2/4B$, and $\rho_c = (A/2B)^3$. Or:

$$k_B T_{c,\max} \approx \left(\frac{\pi}{6}\right)^{2/3} \frac{G^2 m_{\rm H}^{8/3}}{4K_{\rm NR}} M^{4/3}$$

Can then solve for $M_{\rm min}$ needed to have $T_c \geq T_{\rm ignition}$. For a rough estimate, use $T_{\rm ign} = T_{c,\odot}/10 = 1.5 \times 10^6$ K. This gives $M_{\rm min} = 0.05\,M_\odot$, which isn't bad. Real calculations say closer to 0.08 $M_\odot$.

XI.4.2 Maximum Mass

Things get tricky if pressure is from relativistic particles with $\gamma = 4/3$ (nearly unstable). Which will happen if radiation supplies most of the pressure.

$$P_g = \frac{\rho}{\bar{m}} k_B T_c = \beta P_c$$

and

$$P_r = \frac{a}{3} T_c^4 = (1-\beta) P_c$$

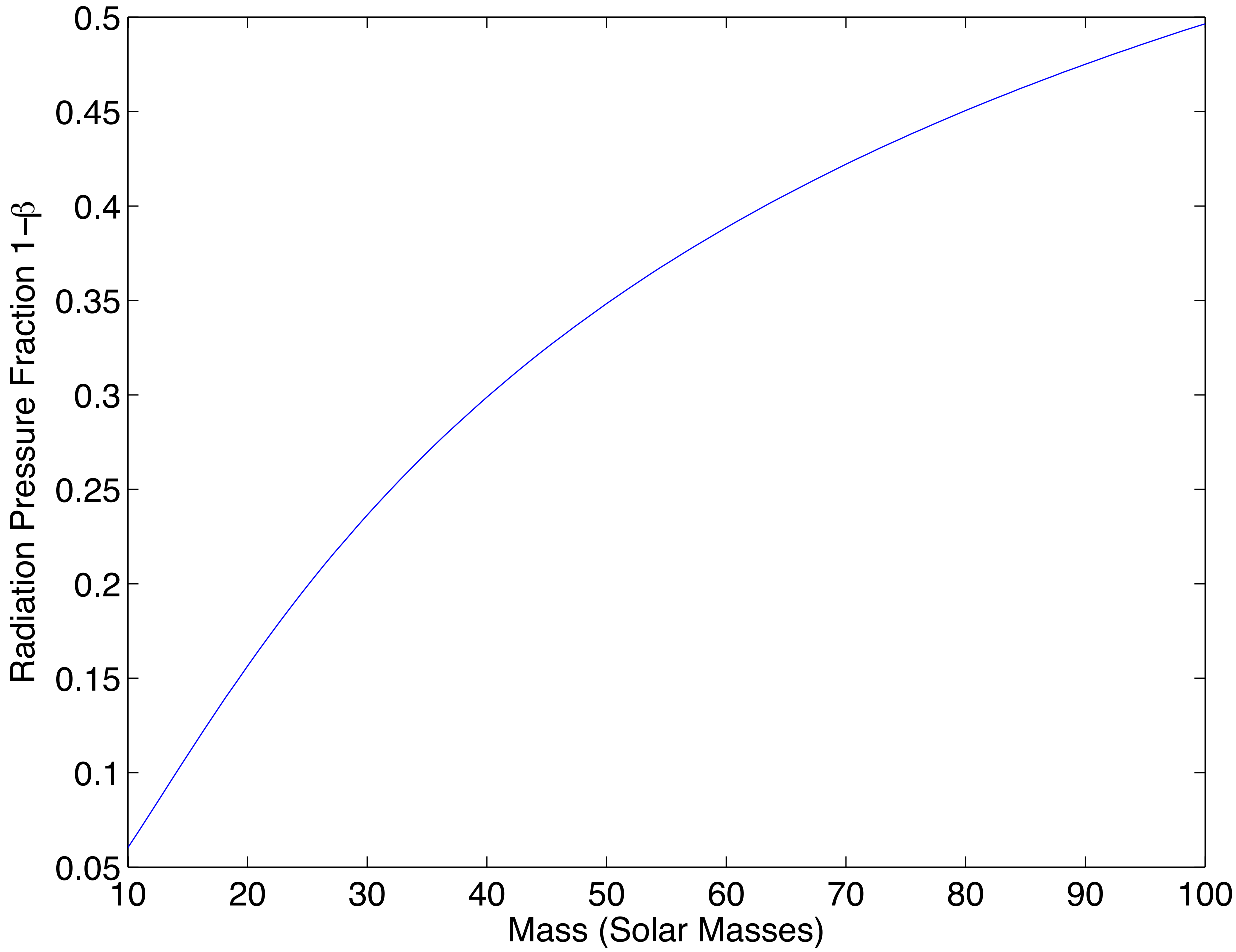
where β is the fraction of total pressure supplied by ions and electrons.

$$P_c = \left(\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B \rho_c}{\bar{m}}\right)^{4/3}$$

Equate this to pressure needed to support the star and get:

$$\left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} = \left(\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B}{\bar{m}}\right)^{4/3}$$

Radiation pressure gets more important as the mass increases.



When $M > 100M_{\odot}$, $1 - \beta > 0.5$ and the star is very unstable. Even $> 50M_{\odot}$ is very rare, but then gain these stars do not live for a long time so they are hard to spot.

XI.4.3 A Fundamental Unit for Stellar Masses

Fine structure constant is measure of EM attraction compared to rest mass. What about the same for gravity for protons? $U = -Gm_{\rm H}^2/r$, with $r = \hbar/m_{\rm H}c$ is the Compton wavelength (reduced). Find:

$$\alpha_G = \frac{Gm_{\rm H}^2}{\hbar c} = 5.9 \times 10^{-39}$$

No units. Compare to $1/137$ for EM: gravity is much weaker. Can write our minimum mass in terms of this (using α_G to get the quantum pieces):

$$M_{\rm min} \approx 16 \left(\frac{k_B T_{\rm ign}}{m_e c^2}\right)^{3/4} \alpha_G^{-3/2} m_{\rm H} \approx 0.03\alpha_G^{-3/2} m_{\rm H}$$

For the max mass do the same, use where $\beta = 0.5$.

$$M_{\rm max} \approx 56\alpha_G^{-3/2} m_{\rm H}$$

Overall, we can say:

$$M_* = \alpha_G^{-3/2} m_{\rm H} = 1.85\, M_\odot$$

Normal stellar life happens near M_*. For things that are much lower or higher it will not be a star or will not live stably. Can also say:

$$N_* = \frac{M_*}{m_{\rm H}} = \alpha_G^{-3/2} = 2 \times 10^{57}$$

is the number of protons in something that can be a star.

XI.5 A Review of Stellar Evolution

XI.5.1 Low-Mass Stars

Main-sequence: core H fusion. If the star is $< 0.5\, M_\odot$ or so, will never fuse helium. May eventually become a red giant (degenerate He core, surrounded by H burning shell, surrounded by puffy envelope) and then a white dwarf, but this can take hundreds of billions of years.

XI.5.2 Middle-Mass Stars

Main sequence lasts for Gyr. Eventually, only He in core. Contracts, increasing pressure and T but not enough for He to ignite. Becomes degenerate. Outside the core H burns in shell. Envelope puffs up, becomes red giant. Ascends the RGB.

H fusion continues to produce He. This "falls" into the core, making it contract further. Eventually, might get He fusion. Since this happens in a degenerate core (when $M < 1.5\, M_\odot$ or so), it will start as an unstable "flash" (raise T does not affect P), but the flash will take a long time to propagate through star so it will not really effect things too much. But extra energy will expand core, making things non-degenerate evetually. Moves to horizontal branch (hotter and smaller). This is basically a He-burning main-sequence, but it is much faster since the reaction is hotter and there is less fuel.

Eventually will exhaust He in core. Contracts again, looks a lot like RGB. Call this phase the AGB. Moves up again, things become somewhat unstable. Pulsations fling off outer layers, lead to PN and WD.

See: `http://www.astronomy.ohio-state.edu/%7Epogge/Lectures/vistas97.html`.

XI.5.3 More Massive Stars

Core never becomes degenerate. So do not become much brighter on the RGB — mostly just become redder. Eventually ignite He, but it is a more gentle process. Move to HB. Exhaust He, then contract again up AGB. Get rid of outer layers, end up as PN and WD.

XI.5.4 Massive Stars

Keep plowing through fusion, making more and more massive elements.

XI.5.5 Schönberg-Chandrasekhar Limit

When does the main-sequence end? Fusion will stop *right* in the core when it becomes mostly He. But the star will still look like a main-sequence star.

If there is no fusion, there is no L generation. So $dT/dr = 0$, and the core is isothermal. How large an isothermal core can you have before the star collapses? That is the S-C limit. Once the collapse occurs, the star changes on K-H timescales (millions of years) instead of nuclear timescale (billions of years).

Star with M, R. Has a core with M_c, R_c, and volume V_c. Core has T_c. P_c is the pressure on the core from all of the bits on top of it.

Start with HSE:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r)$$

Used this before to derive Virial theorem. Multiply both sides by $4\pi r^3$, integrate from center to R_c. More convenient to also use $dM(r)/dr = 4\pi r^2 \rho(r)$ and write:

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$$

with $M_r = M(r)$. This is $4\pi r^3 dP/dM_r = -GM_r/r$. The LHS is:

$$4\pi r^3 \frac{dP}{dM_r} = \frac{d(4\pi r^3 P)}{dM_r} - 12\pi r^2 P \frac{dr}{dM_r} = \frac{d(4\pi r^3 P)}{dM_r} - \frac{3P}{\rho}$$

using $dM(r)/dr = 4\pi r^2\rho(r)$ again. Integrate up to $M_r = M_c$:

$$\int_0^{M_c} dM_r \, \frac{d(4\pi r^3 P)}{dM_r} - \int_0^{M_c} dM_r \, \frac{3P}{\rho} = -\int_0^{M_r} dM_r \, \frac{GM_r}{r}$$

The first term on the LHS is easy, $4\pi R_c^3 P_c$. For the second term, use ideal gas law, $P/\rho = k_B T/\bar{m}_c$ and $T = T_c$ throughout the core. So the integral is $3M_c k_B T_c/\bar{m}_c = 2K_c$ where K_c is the kinetic energy of the core. The RHS is just integrating the gravitational potential energy, so it is U_c. Therefore we get a version of the Virial theorem:

$$4\pi R_c^3 P_c - 2K_c = U_c$$

where we have stopped the integral part of the way out, which is why there is an additional term.

We then take $U_c \approx -(3/5)GM_c^2/R_c$ and substitute back for K_c to get:

$$P_c = \frac{3}{4\pi R_c^3}\left(\frac{M_c k_B T_c}{\bar{m}_c} - \frac{1}{5}\frac{GM_c^2}{R_c}\right)$$

There are two terms: kinetic and gravitational. As the mass increases, the $k_B T_c$ term would increase the pressure, but the other term would decrease it. The pressure will reach a maximum for some value of M_c: this is the maximum pressure which sets the maximum amount of material the core can support outside. Differentiate and solve for the maximum. So this happens at $R_c = (2/5)GM_c\bar{m}_c/k_B T_c$ with a pressure $P_{c,\max} \sim (k_B T_c)^4/M_c^2\bar{m}_c^4$.

But this pressure must support the star, and we know that will be roughly $P \sim GM^2/R^4$ at a temperature $T \sim GM\bar{m}/R$ (here $\bar{m}$ is over the whole star).

So we can find:

$$\frac{M^2}{R^4} \sim \frac{T_c^4}{M_c^2\bar{m}_c^4} \sim \frac{M^4\bar{m}^4}{R^4}\frac{1}{M_c^2\bar{m}_c^4}$$

Or:

$$\frac{M_c}{M} < \alpha\left(\frac{\bar{m}}{\bar{m}_c}\right)^2$$

where α is some constant. Detailed calculations find $\alpha \approx 0.4$, so for $\bar{m}_c = 2\bar{m}$ (He vs. H) $M_c < 0.1M$. Also, increasing $\bar{m}_c$ has a destabilizing effect.

Lecture XII White Dwarfs

Phillips 6.1

XII.2 Sirius B

Star Sirius, brightest in the sky. Found that the star had a proper motion that changed with time, as if from a binary companion. Found companion a while later, Sirius B. From the orbital period, knew that it has a mass of $\sim 0.8\,M_\odot$. But it was much fainter than Sirius A, $\sim 1/360 L_\odot$. It can't be further away (in an orbit). How to resolve this? Cold?

Found temperature from spectrum, looks "white." $T = 8000\,$K. This gives $R \sim 19000\,$km (a factor of 4 different from the true value). Compare to 700,000 km for the Sun. So much much smaller.

Also saw that spectral lines from Sirius B had a redshift relative to Sirius A, once the orbit was corrected. Turns out this is due to general relativity: gravitational redshift. Overall, this is a very small, dense object.

These should be very numerous: the ones that they found first were really nearby, and they are reasonably faint, so there could be many invisible ones out there. In fact they are the end-state of most stars.

XII.3 White Dwarf

End of life for a star of $\sim M_\odot$. We have an inner, isothermal core, mostly of C and O. Surrounded by burning shell of He. Surrounded by burning shell of H. Surrounded by large, puffy envelope. This is AGB phase. Characterized by violent mass loss, leading to planetary nebula.

If the star were more massive could ignite C. But $< 8\,M_\odot$ or so, cannot. So the fuel runs out, the envelope is ejected, and we are left with what is left.

XII.4 Mass and Central Density

Now call this white dwarf. Starts out hot, but as it cools degeneracy pressure from electrons becomes more and more important.

In the center, Y_e is electron fraction: number of electrons per nucleon. $Y_e \approx (1 + X_{\rm H})/2$ ($Y_e = 1$ for H, $1/2$ for most other stuff). With this:

$$n_e = Y_e \frac{\rho_c}{m_{\rm H}}$$

As degeneracy becomes important:

$$P = K_{\rm NR} n_e^{5/3} = K_{\rm NR} \left(\frac{Y_e \rho_c}{m_{\rm H}}\right)^{5/3}$$

Equate to $\left(\frac{\pi}{36}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$ (needed to support a star with high central density), get:

$$\rho_c \approx \frac{3.1}{Y_e^5}\left(\frac{M}{M_*}\right)^2 \frac{m_{\rm H}}{(h/m_e c)^3}$$

with M_* defined from before.

What happens when $p_F c \sim m_e c^2$? Electrons become relativistic, and we use a different EOS. Another way to say it is $n_e > (m_e c/h)^3$. Or, $\rho > m_{\rm H}(m_e c/h)^3$. So what we derived will only be valid if $M < M_*$.

Take C/O WD with $M = 0.4\,M_\odot$. Predict $\rho_c = 4.6 m_{\rm H}(m_e c/h)^3 = 5.4\times10^8\,{\rm kg\,m^{-3}}$ (compare to Pt: $2\times10^4\,{\rm kg\,m^{-3}}$). Electrons have $p_F = 0.65 m_e c$, or $E_F = 0.19 m_e c^2$. So ignoring relativity is not a great approximation.

When we include relativity, the central density ends up higher. In particular, ρ_c goes up faster than M^2. Eventually we get to:

$$P = K_{\rm UR}\left(\frac{Y_e\rho_c}{m_{\rm H}}\right)^{4/3}$$

Equate this to the pressure needed to support the star:

$$K_{\rm UR}\left(\frac{Y_e\rho_c}{m_{\rm H}}\right)^{4/3} = \left(\frac{\pi}{36}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$$

But how can this be true? LHS and RHS have the same power of ρ_c. So obviously there are problems in the math. But is this a real problem?

In fact, we only get a single solution for the mass, which we call the Chandrasekhar mass:

$$M_{\rm Ch} \approx \left(\frac{36}{\pi}\right)^{1/2}\left(\frac{Y_e}{m_{\rm H}}\right)^2\left(\frac{K_{\rm UR}}{G}\right)^{3/2} \approx 2.3 Y_e^2 M_* = 4.3 Y_e^2 M_\odot$$

As the mass of a WD becomes larger, the density increases more and more rapidly. When the star gets to $M_{\rm Ch}$, the density would go to infinity. In reality, there is new physics, but this is a real change in how the star behaves.

We can write a more general model for a WD. Use full relativistic mass/energy:

$$\epsilon^2 = m_e^2 c^4 + p^2 c^2$$

Integrate over phase space to get the pressure, like we did earlier. If we assume things are denegerate, occupancy is 1 for $p < p_F$ and 0 for $p > p_F$.

$$P = \frac{8\pi m_e^4 c^5}{3h^3}\int_0^{x_F} dx\,\frac{x^4}{(1+x)^{1/2}}$$

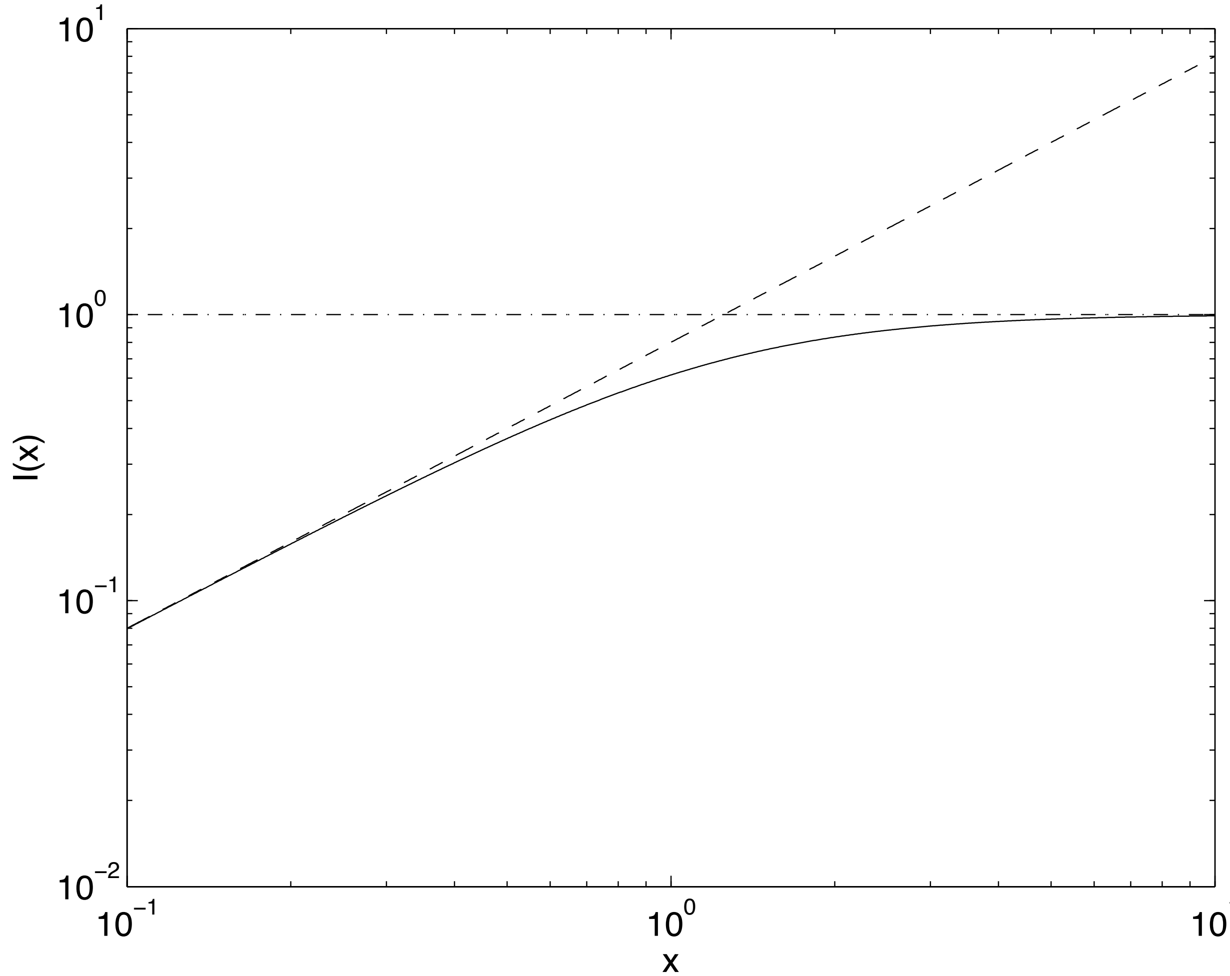
with $x = p/m_e c$. This integration is up to

$$x_F = \frac{p_F}{m_e c} = \left(\frac{3n_e}{8\pi}\right)^{1/3}\frac{h}{m_e c} = \left(\frac{3Y_e\rho_c}{8\pi m_{\rm H}}\right)^{1/3}\frac{h}{m_e c}$$

From this we get:

$$P = K_{\rm UR} n_e^{4/3} I(x_F)$$

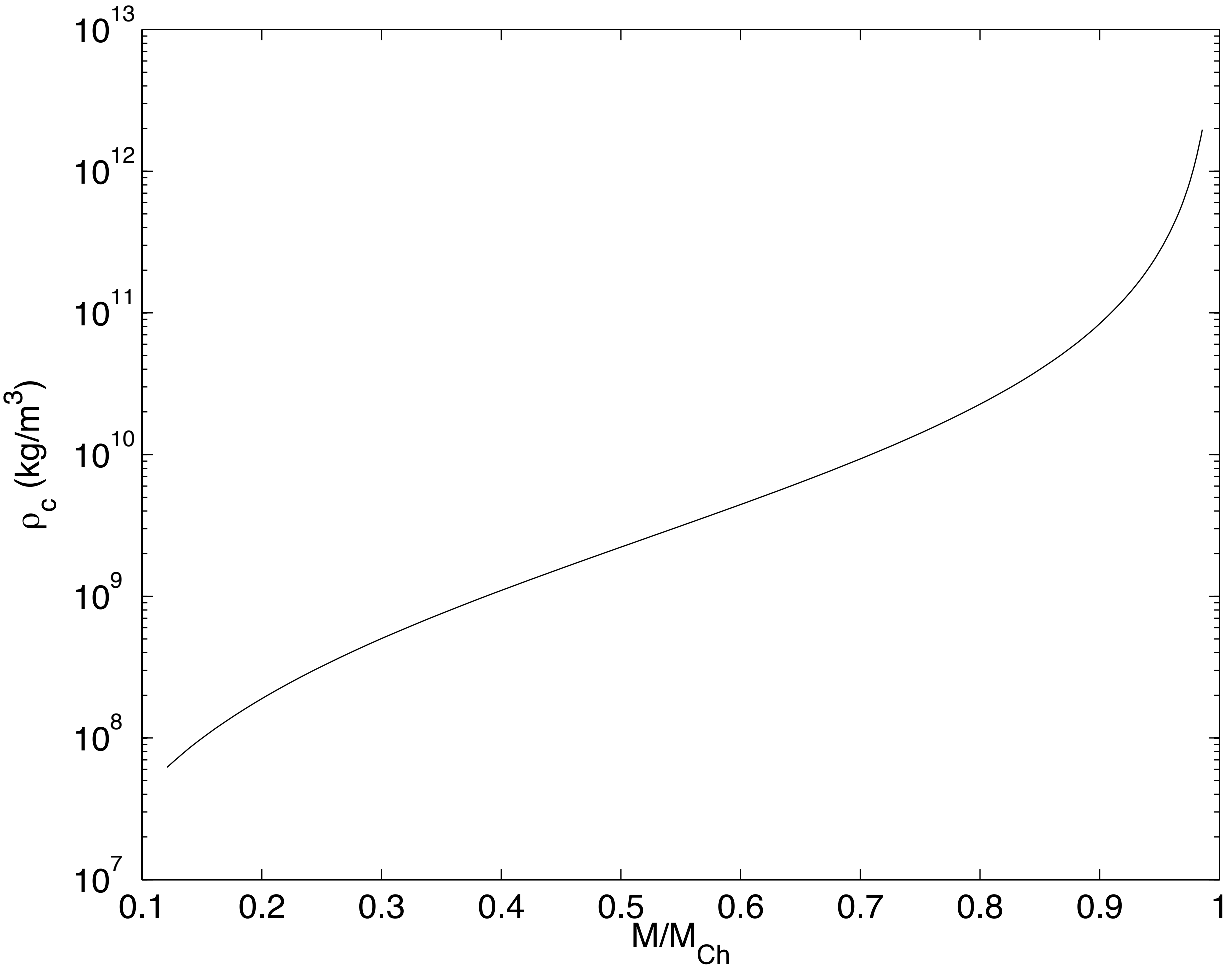
where $I(x)$ is a function that is 1 for $x = \infty$ ($x_F \gg 1$) and goes to $4x/5$ for $x = 0$ ($x_F \ll 1$)



Can equate this with pressure from before, and find:

$$M \approx (I(x_F))^{3/2} M_{\rm Ch}$$

This relates mass to Fermi momentum, which in turn is related to density. Or it relates density to mass. Solve for ρ_c



The density goes up to ∞ when M approaches $M_{\rm Ch}$. In fact a more accurate calculation is similar, but use a polytrope $P \propto \rho^{4/3}$ throughout the star. With this, we find

$$M_{\rm Ch} \approx 5.8 Y_e^2 M_\odot$$

This is a fundamental result realized by Chandrasekhar in 1931. Took him a while to pursuade the rest of the world. A low-mass WD will just cool. But if it has a higher mass the WD stage cannot happen. We will look at this later.

XII.5 Mass and Radius

Density increases as the mass increases, $\rho \sim M^2$. But density $\rho \sim M/R^3$. If these are both true, then R must decrease with mass. THis is unlike what happens for star ($R \sim M$ for stars like the Sun) or planets (balls of rock, $R \sim M^{1/3}$).

If NR, $P \propto \rho^{5/3}$ throughout the star. If this is true, can solve Lane-Emden equation from before, find $\langle\rho\rangle = \rho_c/6$. So we can get:

$$\langle\rho\rangle \approx \frac{0.51}{Y_e^5}\left(\frac{M}{M_*}\right)^2 \frac{m_{\rm H}}{(h/m_e c)^3}$$

and this is $M/(4\pi R^3/3)$, so

$$R \approx 0.77 Y_e^{5/3} \left(\frac{M_*}{M}\right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_e c}$$

So the characteristic size is $\alpha_G^{-1/2}\frac{h}{m_e c} \approx 3\times10^7$ m ($\sim 5R_\oplus$), and the density is $\sim m_{\rm H}/(h/m_e c)^3 \approx 1\times10^8\,{\rm kg\,m^{-3}}$. Or, for $M_\odot$ and $Y_e = 0.5$,

$$R \approx \frac{R_\odot}{74}\left(\frac{M_\odot}{M}\right)^{1/3}$$

This goes *down* as mass goes up. Strange, but consistent with what we see for various WDs.

Use this together with $T_{\rm Eff}$ to get L:

$$L \approx \frac{1}{74^2}\left(\frac{M}{M_\odot}\right)^{-2/3}\left(\frac{T_{\rm Eff}}{6000\,{\rm K}}\right)^4 L_\odot$$

Which agrees with what we find for Sirius B.

Put this together with $T_{\rm Eff}(t)$ from cooling. $L \propto T_{\rm Eff}^4$ so this cools along a particular track in the HR diagram. Since it also depends on mass, but the masses don't change by a very large amount, the luminosities are close together for a range of WDs (lower limit on M from stellar evolution).

Now look at effect of gravity on photons.

$$g = \frac{GM}{R^2} \approx 74^2\left(\frac{M}{M_\odot}\right)^{5/3}\frac{GM_\odot}{R_\odot^2}$$

For photons, think about change in energy from potential with effective mass $m = h\nu/c^2$. So energy is $h\nu_0 - GmM/R$ on the surface, with ν_0 determined from the local atomic physics. When it goes up, needs to overcome potential energy, loses "mass". Ends up going to $r=\infty$, with $\Delta\nu = \nu - \nu_0 = -GM\nu/Rc^2$. Or, $\Delta\nu/\nu = -GM/Rc^2 = -\Delta\lambda/\lambda$. With M/R relation:

$$\frac{\Delta\lambda}{\lambda} \approx 74\left(\frac{M}{M_\odot}\right)^{4/3}\frac{GM_\odot}{R_\odot}$$

How much is this? $\sim 10^{-4}$, or $\sim 0.5\,$Å for optical light. This is measurable.

XII.5.1 Another Derivation for Chandrasekhar Mass

Applies to both WD and NS.

N fermions in a star, can be electrons or neutrons or whatever. Star has radius R, so $n \sim N/R^3$ is density of fermions. The volume for each will be $\sim 1/n$, and from the uncertainty principle $\Delta x\Delta p \sim \hbar$ get $p \sim \hbar n^{1/3}$. Or,

$$E_F \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}$$

The gravitational energy per fermion is:

$$E_G \sim -\frac{GMm_p}{R}$$

since all of the mass is in the protons/baryons, i.e., $M = Nm_p$.

Total energy for a fermion is then:

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{GNm_p^2}{R}$$

Stable equilibrium will be at a minimum of E.

When $E > 0$ (small N), can decrease E by increasing R. This makes E_F smaller, so electrons become less relativistic, with $E_F \sim p_F^2 \sim 1/R^2$. There will be a minimum at a fixed value of R at a stable equilibrium.

But when N large ($E < 0$), can make E more negative by making R smaller. E will effectively get more negative without bound, so there is no stable solution and collapse will occur.

Max N for stability is from setting $E = 0$:

$$N_{\max} \sim \left(\frac{\hbar c}{Gm_p^2}\right)^{3/2} = \alpha_G^{-3/2} = 2 \times 10^{57}$$

As we saw before, this is the fundamental number that determines the mass scales.

The instability comes from $E_F \sim mc^2$. From what we have above,

$$R \sim \frac{\hbar}{mc}\left(\frac{\hbar c}{Gm_p^2}\right)^{1/2}$$

where we have not specified what the fermion is. So we can do this for electrons (WD) or neutrons (NS). Which gives the ratio in the radii as $\sim (m_p/m_e)$. $\sim 5 \times 10^6$ m for WD, $\sim 3 \times 10^3$ m for NS.

XII.5.2 What about a Minimum Mass?

As M increases, R decreases and ρ_c increases until near limit. Then they hit 0 and ∞. But at the other end?

Keep electrons degenerate, for simpicity.

$$k_B T \ll E_F = \frac{1}{2m_e}\left(3\pi^2\hbar^3 n_e\right)^{2/3} = 26(Y_e\rho_3)^{2/3}\,\mathrm{eV}$$

with $\rho_3 = \rho/10^3\,\mathrm{kg\,m^{-3}}$ is the density of water. Or, $T \ll 3 \times 10^5 (Y_e\rho_3)^{2/3}$ K, and degeneracy will still dominate. But what breaks down?

We assume free electrons to get $P \propto \rho^{5/3}$. But the Coulomb interactions electrons/ions and electron/electron will start to become important. Compare E_F and E_C. Take a single box around an ion (“Wigner-Seitz cell”) with Z electrons. We have $n_i = \rho/Am_p$ and $n_e = Zn_i$. The radius of the cell is r_i, and each electron will have its own little volume $r_S = Z^{-1/3} r_i$.

If we use the uncertainty principle for each electron, it has momentum $\hbar/r_s$. Or, energy $(\hbar/r_s)^2/2m_e$ (NR). Then the total kinetic energy for Z electrons is:

$$E_K \sim Z \left(\frac{\hbar}{r_s}\right)^2 \frac{1}{2m_e}$$

And the Coulumb energy is:

$$E_C \sim -\frac{Z^2 e^2}{r_i} = -\frac{Z^{5/3} e^2}{r_s}$$

When r_s is very small, $|E_K| \gg |E_C|$ and we can ignore Coulomb forces. THis happens for high density. But when r_s is bigger we cannot do that.

$$\left|\frac{E_K}{E_C}\right| \sim Z \frac{\hbar^2}{r_s^2} \frac{1}{m_e} \frac{r_s}{Z^{5/3} e^2} = Z^{-2/3} \left(\frac{\frac{\hbar^2}{m_e e^2}}{r_s}\right) = Z^{-2/3} \frac{a_0}{r_s}$$

With a_0 the Bohr radius. So when $Z^{2/3} r_s > a_0$ Coulomb forces are important, or when:

$$\rho = \frac{Am_p}{4\pi r_i^3/3} < ZA \frac{m_p}{4\pi a_0^3/3} \approx 3ZA \times 10^3 \,\mathrm{kg\,m^{-3}}$$

We can define $\rho_{\mathrm{atom}} = m_p/(4\pi a_0^3/3)$, and the limit is then $\rho_{\mathrm{Coul}} = ZA\rho_{\mathrm{atom}}$.

As M drops, P drops and ρ drops. When ρ gets near ρ_{Coul}, different physics takes over. We end up with uncompressed matter, held up by electrostatics. We call this a planet! This is at effectively constant density, independent of mass. At what mass?

$$R \sim Y_e^{5/3} \alpha_G^{-1/2} \left(\frac{M_*}{M}\right)^{1/3} \frac{\hbar}{m_e c}$$

So we can write

$$\rho \sim \frac{3M}{4\pi Y_e^{5/3} \alpha_G^{-1/2} \left(\frac{M_*}{M}\right)^{1/3} \frac{\hbar}{m_e c}} \sim \rho_{\mathrm{Coul}}$$

with $\rho_{\mathrm{Coul}} = ZAm_p/(4\pi a_0^3/3)$ and $a_0 = \hbar/m_e c\alpha$. So

$$M_P \sim (ZAY_e^5)^{1/2} (\alpha^{3/2} M_*)$$

with $\alpha^{3/2} M_* \approx 1.15 \times 10^{-3}\, M_\odot \approx M_{\mathrm{Jupiter}}$

Lecture XIII Core Collapse

Phillips 6.2

Star with $> 10\,M_\odot$ (or so). Will go through all stages of nuclear burning in $< 10\,\mathrm{Myr}$. Eventually have Si burning making iron at $T = 3\times10^9$ K, surrounded by shells of lighter elements. Cannot get energy out of iron via fusion, so core contracts (just like RGB). Stabilized somewhat by degenerate electrons, but Si burning dumps increasing amounts of stuff on and electrons get increasingly relativistic. When the core is at $M_{\mathrm{Ch}} \approx 1.4\,M_\odot$, electrons have become ultra-relativistic and the core can no longer support itself.

XIII.1.3 Onset of Collapse

During contraction T rises. If makes exothermic reactions possible, then T and pressure rise and collapse stops. But what if no exothermic reaction is possible? If only endothermic, reduces P, makes contraction into collapse.

Possible reactions are photodisintigration of nuclei and electron capture (inverse β decay). Photodisintigration: KE is used to unbind nuclei. Electron capture: KE of electrons is converted into KE of neutrinos (and lost). These both suck up energy very effectively, turning contraction into free-fall. At this point $\rho \approx 10^{12}\,\mathrm{kg\,m^{-3}}$, and free-fall happens with:

$$\tau_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32G\rho}} \approx 1\,ms$$

XIII.1.4 Photodisintigration

T rises enough such that photons have nuclear-scale energies. Takes a tightly-bound Fe nucleus and makes two or more loosely bound nuclei, absorbing binding energy. This can take many paths, but as an example:

$$\gamma +^{56}\mathrm{Fe} \leftrightarrow 13^4\mathrm{He} + 4n$$

equilibrium between iron and helium + neutrons. This takes:

$$Q = (13m_4 + 4m_1 - m_{56})c^2 = 124.4\,\mathrm{MeV}$$

So $1\,\mathrm{kg}$ of Fe can absorb 2×10^{14} J ($50\,\mathrm{kton}$ of TNT). Use Saha equation to determine relative fractions:

$$\mu_{56} = 13\mu_4 + 4\mu_1$$

with

$$\mu_A = m_Ac^2 - k_BT\ln\left(\frac{g_An_{Q,A}}{n_A}\right)$$

and

$$n_{Q,A} = \left(\frac{2\pi m_Ak_BT}{h^2}\right)^{3/2}$$

which gives:

$$\frac{n_4^{13} n_1^4}{n_{56}} = \frac{g_4^{13} g_1^4}{g_{56}} \frac{n_{Q,4}^{13} n_{Q,1}^4}{n_{Q,56}} e^{-Q/k_B T}$$

The g factors can be complicated, but we will assume $g_1 = 2$, $g_4 = g_{56} = 1$. From this we get that roughly 75% of the Fe is dissociated when $\rho = 10^{12}\,\mathrm{kg\,m^{-3}}$ and $T = 10^{10}\,\mathrm{K}$.

For higher temperatures still:

$$\gamma +^4 \mathrm{He} \leftrightarrow 2p + 2n$$

Overall, in collapse of $1.4\,M_\odot$, absorb 4×10^{44} J in breaking Fe and 1×10^{45} J in breaking He, for a total of $E_{\rm photo} \approx 1.4 \times 10^{45}$ J. This is $L_\odot \times 10^{11}$ yr. Easy to see how this could lead to collapse.

XIII.1.5 Electron Captures

Neutron can decay on its own (β decay):

$$n \to p + e^- + \bar{\nu}_e$$

with half-life of 10.25 min. This produces electrons and neutrinos with total energy of $(m_p - m_n)c^2 = 1.3\,\mathrm{MeV}$, so the max electron energy is $1.3\,\mathrm{MeV}$. If electrons with that energy cannot be produces, neutrons cannot decay. For instance, if all of the low-energy spots are filled by other electrons (in a dense gas of degenerate electrons with $E_F > 1.3\,\mathrm{MeV}$) this cannot happen.

Moreover, if electrons with $E > 1.3\,\mathrm{MeV}$ are around, they can capture onto protons to form neutrons:

$$e^- + p \to n + \nu_e$$

This can happen even if the protons are in nuclei. For instance, *neutronization* starts when:

$$e +^{56} \mathrm{Fe} \to^{56} \mathrm{Mn} + \nu_e$$

is favorable, at $\rho > 1.2 \times 10^{12}\,\mathrm{kg\,m^{-3}}$. This happens when $E_F = m_e c^2 + 3.7\,\mathrm{MeV}$. The Mn would normally decay back in 2.6 hr, but here instead it will capture again to make ^{56}Cr. And so on as the density goes up past $10^{13}\,\mathrm{kg\,m^{-3}}$.

This speeds up further when $\rho > 10^{14}\,\mathrm{kg\,m^{-3}}$. Almost all of the energy in neutrinos is lost. So the pressure support goes away quickly. How much energy? Core has $\sim 10^{57}$ electrons, which could make 10^{57} neutrinos. Each capture will take an electron with $E \approx 10\,\mathrm{MeV}$, appropriate for $\rho > 2 \times 10^{13}\,\mathrm{kg\,m^{-3}}$. So total energy is:

$$E_{\rm cap} \approx 10^{57} \times 10\,\mathrm{MeV} = 1.6 \times 10^{45}\,\mathrm{J}$$

which is similar to that from photodisintigration. But in this case it is carried from the star in a burst of neutrinos. If they could get out immediately, the burst would take ∼ms. But in fact when the core density is $> 10^{14}\,\mathrm{kg\,m^{-3}}$ the mfp becomes comparable to the size of the core, a few km. They will get out, but it will take a few seconds.

XIII.1.6 And Then...

The collapse will proceed on the free-fall timescale. What will stop it? It will stop when the bulk density is comparable to the nuclear density. For a nucleus with A nucleons, $R \approx r_0 A^{1/3}$ with $r_0 = 1.2\,\mathrm{fm}$. So $\rho_{\mathrm{nuc}} = 3m_n/4\pi r_0^3 = 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}}$. Once we are that this stage we need new physics (neutron degeneracy, nuclear forces). The collapse will stop when the density is a few times this as strong nuclear force comes in, and creates a "bounce". This propels a shock wave through the material, leading to a supernova.

Supernovae are observed to have 10^{44} J of KE and 10^{42} J of optical energy (over the first few years). Where does this come from? Gravitational binding energy:

$$E_G \sim \frac{GM^2}{R_{\mathrm{core}}} = 3 \times 10^{46}\,\mathrm{J} \left(\frac{M}{M_\odot}\right)^2 \frac{10\,\mathrm{km}}{R}$$

Which is orders of magnitude more than we see. We only see a small fraction of this, and we don't quite know exactly how the energy is partitioned. But this is plenty of energy compared to photodisintigration or electron capture. Most of the energy in fact comes via neutrinos, either right during collapse or later, as the neutron star cools. This happens over the diffusion timescale, $R^2/c\bar{l}$, and each flavor of neutrinos will carry $\sim E_G/6$.

On Feb 23, 1987, two neutrino detectors recorded excesses. They identify neutrinos via:

$$\bar{\nu}_e + p \to n + e^+$$

and if the positron has enough energy, it will be faster than the local speed of light in water, so it will emit Čerenkov radiation. That can be detected. Only ~ 1 in 10^{15} neutrinos is expected to be detected.

Saw about 20 neutrinos over $\sim 10\,\mathrm{s}$. Expect that this is the diffusion timescale, which makes sense if $R \approx 100\,\mathrm{km}$ and $\bar{l} = 10^{-4}R$. This came from SN 1987A in the LMC, implying an energy of about $(0.3 - 0.5) \times 10^{46}$ J for $\bar{\nu}_e$.

Energies of the neutrinos is consistent with $T_{\mathrm{Eff}} \approx 5 \times 10^{10}\,\mathrm{K}$. Compare to internal temperature, using the mfp:

$$T_{\mathrm{Eff}} \approx \left(\frac{\bar{l}}{R}\right)^{1/4} T_I$$

implies an internal temperature of $10^{11-12}\,\mathrm{K}$.

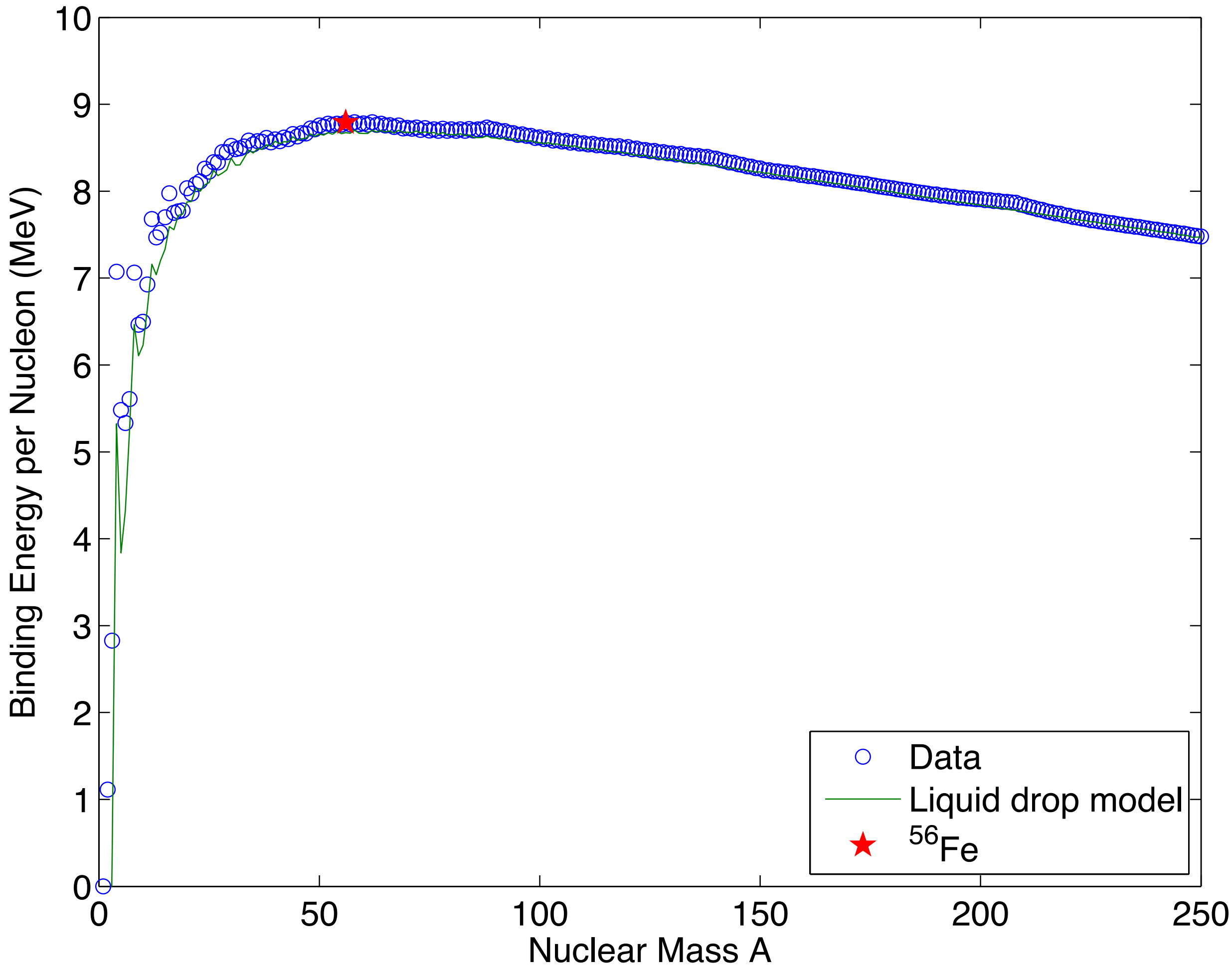
Lecture XIV Neutron Stars

Phillips 6.3

A NS is born after core-collapse as a hot, degenerate object. Typical internal temperature is 10^{11-12} K. Cools (primarily via neutrinos) to 10^9 K in a day and 10^8 K in 100 yr. Even though they appear high, are still "cold" in that $k_B T \ll E_F$.

XIV.1.7 Matter inside a NS

Stability of nuclei: peaks near ^{56}Fe. For smaller nuclei, too many nucleons are near the surface, and for more massive nuclei the Coulomb repulsion matters more (remember the liquid drop model):



But this only works in isolation. With relativistic electrons present the equilibrium is different. They have enough energy for inverse beta decay. This turns protons into neutrons, and we end up with nuclei with many many neutrons (heavier than Fe). For example, with $\rho \sim 10^{14}\,\mathrm{kg\,m^{-3}}$, we might have ^{76}Fe and ^{78}Ni.

This keeps going as the material gets cooler and denser. Eventually, around $4 \times 10^{14}\,\mathrm{kg\,m^{-3}}$, we get *neutron drip*. Neutrons leak out of nuclei, and we get a new equilibrium with heavy nuclei, neutrons, and electrons. We can calculate this OK for $\rho < \rho_{\mathrm{nuc}} = 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}}$. Beyond this density we do not understand the many-body interactions that occur, and it gets even more uncertain beyond $10^{18}\,\mathrm{kg\,m^{-3}}$ (pions, muons, hyperons, ...).

Why are free neutrons OK? For most neutrons, cannot have beta decay since electrons are blocked from forming by Pauli exclusion principle: $n \to p + e^- + \bar{\nu}_e$ cannot happen freely. This means that all neutrons with energies below $E_F(n)$ will be blocked from decaying if:

$$E_F(n) < E_F(p) + E_F(e)$$

since the proton and electron cannot form. But the neutron can decay if

$$E_F(n) > E_F(p) + E_F(e)$$

Overall, there will be an equilibrium, and it is characterized by:

$$E_F(n) = E_F(p) + E_F(e)$$

(You can also get this by noting that chemical potential μ for $T = 0$ is just the Fermi energy, and $k_B T \ll E_F$ for T is effectively 0).

So we can have reactions when the Fermi energies balance, with:

$$n \to p + e^- + \bar{\nu}_e$$

$$p + e^- \to n + \nu_e$$

happening (note that the neutrinos escape). We use the relation between Fermi momentum and density:

$$p_F = h\left(\frac{3n}{8\pi}\right)^{1/3}$$

For n and p, they are NR, so:

$$E_F(n) = m_n c^2 + \frac{p_F(n)^2}{2m_n}$$

and the same for the protons. Electrons are relativistic, so

$$E_F(e) = p_F(e)c$$

We also require $n(e) = n(p)$ for neutrality. So we get:

$$hc\left(\frac{3n_p}{8\pi}\right)^{1/3} + \left(\frac{3n_p}{8\pi}\right)^{1/3}\frac{h^2}{2m_p} - \left(\frac{3n_n}{8\pi}\right)^{1/3}\frac{h^2}{2m_n} \approx m_n c^2 - m_p c^2$$

We can use $(m_n - m_p)c^2 = 1.3\,\mathrm{MeV}$ and solve this. For example, at $\rho = 2 \times 10^{17}\,\mathrm{kg\,m^{-3}}$, $n_n = 1 \times 10^{44}\,\mathrm{m^{-3}}$ and $n_e = n_p = n_n/200$, so there is 1 proton for 200 neutrons. This is very far from normal matter.

XIV.1.8 Size of a NS

Analogous to WD, treat a NS as a uniform fluid of degenerate, NR neutrons. So

$$n_n \approx \frac{\rho_c}{m_n}$$

depending on the central density. We can modify the WD equations to work for neutrons, with $Y_e \to Y_n = 1$ and $m_e \to m_n = m_p$. So

$$\rho_c \approx 3.1 \left(\frac{M}{M_*}\right)^2 \frac{m_n}{(h/m_n c)^3}$$

Again, the relevant density scale is $m_n/(h/m_n c)^3$. Now,

$$R \approx 0.77 \left(\frac{M_*}{M}\right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_n c}$$

So we have $\alpha_G^{-1/2} h/m_n c$ as the fundamental scale, which gives about 17 km. As expected, this is about 2000 times smaller than a WD computed the same way.

However, doing this for a NS properly is much harder. The neutrons will be somewhat relativistic when $M \sim M_*$, which we do have (measured masses are $1.1 - 2.0\, M_\odot = 0.6 - 1.1\, M_*$). There are complicated, many-body interactions that we cannot do well, and other states of matter that might be important.

Another issue is GR: we cannot use Newtonian gravity for HSE to compute the structure. Compare GMm/R to mc^2, gravitational to rest-mass energy:

$$\frac{GM}{Rc^2} \approx 0.2 \left(\frac{M}{M_*}\right)^{4/3}$$

so GR is important. But we still have a useful estimate.

XIV.1.8.1 Binding Energy

Gravitational binding energy is roughly comparable to that released as neutrinos during core-collapse.

$$E_B \approx \frac{GM^2}{R} \approx 0.2 \left(\frac{M}{M_*}\right)^{7/3} M_* c^2 = 7 \times 10^{46}\,\mathrm{J} \left(\frac{M}{M_*}\right)^{7/3}$$

XIV.1.9 Interior Structure

Transition Density ($\mathrm{kg\,m^{-3}}$)	Composition	Degeneracy pressure
	iron nuclei, NR electrons	electrons
$\approx 1 \times 10^{9}$	electrons become relativistic	
	iron nuclei, relativistic electrons	electrons
$\approx 1 \times 10^{12}$	neutronization	
	neutron-rich nuclei, relativistic electrons	electrons
$\approx 4 \times 10^{14}$	neutron drip	
	neutron-rich nuclei, free neutrons, relativistic electrons	electrons
$\approx 4 \times 10^{15}$	neutron degeneracy pressure dominates	
	neutron-rich nuclei, free neutrons, relativistic electrons	neutrons
$\approx 2 \times 10^{17}$	nuclei dissolve	
	superfluid neutrons, superconducting protons, relativistic electrons	neutrons
$\approx 4 \times 10^{17}$	pion(?) production	
	superfluid neutrons, superconducting protons, relativistic electrons, other particles?	neutrons

Way down inside crazy stuff happens. For instance, we can get superfluid neutrons: no viscosity or friction to flowing. Can spin forever. And superconductor: no resistance to current. The inside can have other particles present at very high densities, like pions:

$$n \rightarrow p + \pi^{-}$$

which can occur at $\rho > 2\rho_{\rm nuc}$ ($m_\pi \approx 300 m_e \ll m_n$, so mass is still conserved).

The various layers of a NS are:

1. Crust (500 m): heavy nuclei, either in a "fluid" or "lattice." This is along with relativistic degenerate electrons. Mostly Fe at the surface, moving to heavier, more neutron-rich nuclei further down until neutron drip ($4 \times 10^{14}\,\mathrm{kg\,m^{-3}}$).

2. Inner crust (1 km): lattice of nuclei, superfluid neutrons, electrons. This goes up to $\rho_{\rm nuc}$, where the nuclei dissolve and we can't speak of individual nuclei any more

3. Interior: superfluid n, with a smaller amount of superconducting p and e (UR, degenerate).
4. Core? Could be or not. Might be some weird state of matter, or just a rather dense bit of the interior. $\rho_c \approx 10^{18}\,\mathrm{kg\,m^{-3}}$.

Draw shells

XIV.1.10 Maximum Mass

Just like in a WD, when the neutrons get too relativistic we have a $P \propto \rho^{4/3}$ polytrope which is unstable. So there is an effective Chandrasekhar limit for neutron stars, although calculating it precisely is not so easy. We need to worry about the interactions (which we ignored for the WD) and about GR.

If we just use the WD relations and substitute m_n for m_e (along with $Y_n = 1$), we find $M_{\max} \approx 3.1 M_* = 5.8\,M_\odot$. This is not unreasonable, but a bit high.

What about the complications? Interactions between neutrons are attractive at moderate distances, but repulsive very close together. So as you squeeze the NS and pile more mass on, the n-n interactions would tend to make it more stiff and resist collapse. However, if the neutrons can form other particles (pions, etc.) there is less pressure from degenerate neutrons, and the pressure from the new particles is small (remember that $k_BT \ll E_F$). So new particles make it softer. Overall, the interactions tend to increase the maximum mass of the NS.

GR will have the opposite effect. First, we need to be careful about which "mass" we are discussing. The gravitational binding energy is comparable to Mc^2 (or the rest mass). Just like KE in special relativity, in GR we need to account for the gravitational energy. Gravity also gets effectively stronger at very high densities (the gravitational energy creates gravity), which decreases the maximum mass. We can see this through the TOV equation, which replaces HSE in GR:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\frac{(1+P/\rho c^2)(1+4\pi r^3 P/mc^2)}{1-2Gm/rc^2}$$

All forms of energy are counted when adding up the sources of gravity. Both the gravitational field and the pressure are sources of energy, and they create gravity. So pressure is on both the left and right sides of this equation. Which means that if you put in more pressure to reduce gravity, it will end up just creating more gravity!

Solving this for non-interacting neutrons gives $0.7\,M_\odot$ for the maximum mass.

Let us look at this for a constant density model, $\rho = \rho_0$ (incompressible, polytrope with $n = 0$). The solution to just HSE (Lane-Emden equation) without GR is:

$$P(r) = \frac{2\pi G}{3}\rho_0^2(R^2 - r^2)$$

At the center, the pressure is:

$$P_c = P(0) = \frac{2\pi G}{3}\rho_0^2 R^2 = \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_0^{4/3}$$

This pressure is finite for all masses, so in Newtonian gravity the mass can be whatever you'd like.

In GR, we can also solve the corresponding equation (messier), and find:

$$P = \rho_0 c^2 \left[\frac{(1 - 2GMr^2/R^3c^2)^{1/2} - (1 - 2GM/Rc^2)^{1/2}}{3(1 - 2GM/Rc^2)^{1/2} - (1 - 2GMr^2/R^3c^2)^{1/2}} \right]$$

which gives:

$$P_c = \rho_0 c^2 \left[\frac{1 - (1 - 2GM/Rc^2)^{1/2}}{3(1 - 2GM/Rc^2)^{1/2} - 1} \right]$$

The pressure will only be finite if $GM/Rc^2 < 4/9$. And we can subsitute in for the densities etc. But let's write:

$$\rho_0 = \frac{3m_n}{4\pi r_n^3}$$

as the density of a neutron, with

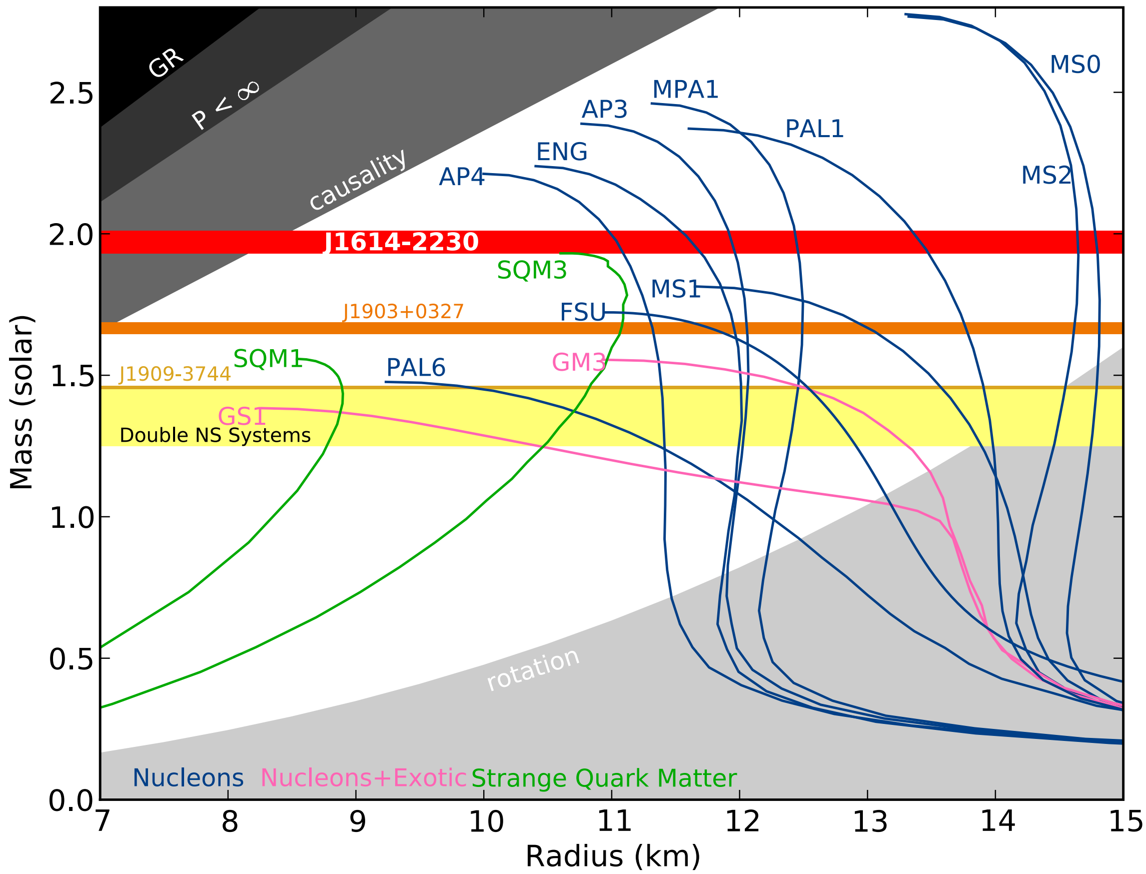
$$r_n = f_n \frac{h}{m_c c}$$

So r_n is the effective radius. $h/m_n c$ is the Compton wavelength, and f_n is some dimensionless parameter that allows us to adjust things. For instance, $\rho_{\rm nuc} = 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}}$ corresponds to $f_n = 0.9$. So we can write our condition for a finite central pressure to be:

$$M_{\rm max} = \left(\frac{8\pi f_n}{9} \right)^{3/2} M_*$$

This is smaller than before, but still M_* is the important number. And this is for matter that *cannot compress*; for real matter the maximum mass will be less, since the pressure needed would be infinite.

Overall, we are not sure what the maximum mass is. It is somewhere between 2 and 5 $M_\odot$. We know that we see NS's with $M = 2\,M_\odot$, and that's important since it tells us something about what they are made of inside: they cannot be made of stuff that would have a maximum mass less than that.



XIV.1.11 Rotation

Conserve angular momentum. Everything has some. Sun rotates ~ 1/month.

Go from core (WD) to NS. We have:

$$\frac{R_{\rm core}}{R_{\rm NS}} \approx \frac{m_n}{m_e}\left(\frac{Z}{A}\right)^{5/3} = 512$$

for $Z/A = 26/56$ for iron. This gives us the ratio of sizes. We can then conserve angular momentum (assuming mass is the same), $L = I\omega$ with $I = CMR^2$ (C depends on internal structure).

$$I_{\rm core}\omega_{\rm core} = I_{\rm NS}\omega_{\rm NS}$$

which gives

$$\omega_{\rm NS} = \omega_{\rm core}\left(\frac{R_{\rm core}}{R_{\rm NS}}\right)^2$$

Or, in terms of period P,

$$P_{\rm core} = P_{\rm wd}\left(\frac{R_{\rm NS}}{R_{\rm core}}\right)^2 \approx 4\times 10^{-6} P_{\rm core}$$

If we use $P_{\rm core} = 1350\,{\rm s}$ for a known WD, we find periods of a few ms for a NS.

How does this compare with the minimum period? Use what we derived before, which can also be expressed as:

$$\frac{GM}{R^2} = \omega_{\rm max}^2 R$$

which gives

$$P_{\rm min} = \frac{2\pi}{\omega_{\rm max}} = 2\pi \left(\frac{R^3}{GM}\right)^{1/2} = 0.6\,{\rm ms}\left(\frac{M_*}{M}\right)$$

So as long as the period is $> 1\,{\rm ms}$, we don't have to worry about the NS breaking up.

XIV.1.12 Magnetic Field

Just as the angular momentum is conserved, so will be magnetic flux. This is because the material is a very good conductor. Roughly, $\pi R^2 B$ is a constant, so:

$$B_{\rm NS} \approx B_{\rm core}\left(\frac{R_{\rm core}}{R_{\rm NS}}\right)^2 = 1.3\times 10^{10}\,{\rm T}$$

based on the most extreme WD field of $5\times 10^4\,{\rm T}$ (more typical is $10\,{\rm T}$, and $2\times 10^{-4}\,{\rm T}$ for the Sun). So the B could be really high, although it is usually a little more modest (10^8 T).

Lecture XV Pulsars

Neutron stars were predicted in 1934 (neutrons discovered in 1932) and thought to be associated with SNe, but not seen for > 30 yr.

Jocelyn Bell was looking at radio emission over time. Studying scintillation, or flickering, which is irregular changes. Saw that there were little blips that appeared regularly, every 1.337 s. This came from the same part of the sky, which rose at a different time every night (*not* from Earth). These are *pulsars*.

Nature of pulsars was deduced by Tommy Gold. Based on facts:

- Periods of ~ 1 s common, although the Crab pulsar has $P = 33$ ms
- Periods are very stable, change by $\dot{P} \sim 10^{-15}$ s/s

Models:

Binary Star Would need P from orbital period. Can use Kepler's third law to show that for $M = 1\,M_\odot$, $a = 1.6 \times 10^6$ m. Compare to $R = 7 \times 10^8$ m for the Sun, or 5×10^6 m for Sirius B.

Could be orbiting NSs. However, GR says that orbit should gradually grow closer, which period is observed to grow longer.

Pulsating Star WDs oscillate with $P = 100 - 1000$ s. Can show that $P \sim 1/\sqrt{G\rho}$. If were a pulsating NS, then would have period of $\sim 10^{-4}$ times that of WD (ρ is 10^8 times higher), so this is OK, but maybe too short for slow pulsars.

Rotating Star As we saw, can rotate at ~ 1 ms and be OK for a NS. If were WD, would not periods of ~ 10 s at minimum, so can't work.

Result is that pulsars are rotating NSs.

XV.1.13 Crab Pulsar

Center of Crab SNR, from 1054 AD. See $P = 33$ ms, with $\dot{P} = 4.2 \times 10^{-13}$ s/s. What does this imply about change in rotational energy? $I = \frac{2}{5}MR^2$ for uniform sphere. For NS,

$$I \approx 0.24 \left(\frac{M}{M_*}\right)^{1/3} \alpha_G^{-5/2} m_n \left(\frac{h}{m_n c}\right)^2 = 2.5 \times 10^{38}\,\mathrm{kg\,m^2} \left(\frac{M}{M_*}\right)^{1/3}$$

See $\dot{P}$. $\omega = 2\pi/P$, so $\dot{\omega} = -2\pi\dot{P}/P^2 = -2.4 \times 10^{-9}\,\mathrm{s}^{-2}$ (increase by a ms every 90 yr). Use $E_\mathrm{rot} = \frac{1}{2}I\omega^2$, so identify:

$$\frac{dE_\mathrm{rot}}{dt} = I\omega\frac{d\omega}{dt} = -4\pi^2 I\frac{\dot{P}}{P^3}$$

If we take $I = 10^{38} I_{38}\,\mathrm{kg\,m^2}$, have 4.6×10^{31} W. But we can also measure the amount of energy radiated away by the Crab Nebula and find 5×10^{31} W. These balance, and the energy of the Crab Nebula is supplied by the slowing rotation.

Overall, pulsars were found to be rotating neutron stars. We see blips when the "lighthouse" beam crosses the Earth **show animation**. The majority of the energy from the spin-down is invisible: the radio blips are a tiny fraction of the energy.

It is the strong magnetic field that makes this happen.

XV.1.14 Magnetic Dipole Model

Light cylinder: where v to go around is c. We take the magnetic field to be a dipole, $B(r) = B_0(r/R)^{-3}$. A changing magnet releases electromagnetic power per unit area S (*Poynting flux*) $\sim cB^2/\mu_0$. We can roughly relate the spin-down energy loss $I\omega\dot{\omega}$ to the Poynting flux through the light cylinder:

$$4\pi R_{\mathrm{LC}}^2 S_{\mathrm{LC}} \approx I\omega\dot{\omega}$$

with $\omega = 2\pi/P$, $\dot{\omega} = -2\pi\dot{P}/P^2$. $R_{\mathrm{LC}} = cP/2\pi$, so $S_{\mathrm{LC}} = (c/\mu_0)B_{\mathrm{LC}}^2 = (c/\mu_0)B_0^2 R^6/R_{\mathrm{LC}}^2$. So we have:

$$4\pi R_{\mathrm{LC}}^2 \frac{c}{\mu_0} B_0^2 \frac{R^6}{R_{\mathrm{LC}}^6} = 4\pi R^6 \frac{c}{\mu_0} B_0^2 \left(\frac{cP}{2\pi}\right)^{-4} \sim \frac{R^6}{\mu_0}\frac{B_0^2}{c^3} P^{-4} \sim I\frac{\dot{P}}{P^3}$$

This gives:

$$B_0^2 \sim \frac{c^3 \mu_0 I}{R^6} P\dot{P}$$

So from the spin period and the rate at which it is slowing down, we can determine what the magnetic field is!

We can then use this (assuming B =constant) to get $P(t)$. The equation above is a simple ODE. We assume that the spin-down is constant, and have $P(0) = P_0$. Solve:

$$\frac{dP}{dt} = \frac{A}{P(t)}$$

With the solution

$$P(t) = \sqrt{2At + P_0^2}$$

with

$$A = \frac{R^6 B_0^2}{\mu_0 c^3 I} = \dot{P}_{\mathrm{now}} P_{\mathrm{now}}$$

If we assume that $P_{\mathrm{now}} \gg P_0$, then $P(t) \approx \sqrt{2At}$. Alternatively,

$$t = \frac{1}{2A}\left(P(t)^2 - P_0^2\right)$$

We find that the age is $\tau \approx P/2/\dot{P}$, so we also get the age of the system from P and $\dot{P}$. Do this for the Crab pulsar get 1250 years, which is very close to the true age of about 950 years (since people saw the supernova).

P-$\dot{P}$ diagram: HR diagram for pulsars. **draw**. Move through the diagram from upper left to lower right until you die from low voltage (don't actually die, just shut off). This takes 10^{7-8} yrs to get to $P = 10\,\mathrm{s}$ from a typical starting point of $10\,\mathrm{ms}$. Usually born with 10^8 T, but there is a range.

Millisecond pulsars: $P = 1.56\,\mathrm{ms}$, $\dot{P} = 1.1 \times 10^{-19}\,\mathrm{s/s}$. This gives $B = 9 \times 10^4$ T, so much smaller than normal pulsar. And age of 2.3×10^8 yr. Cannot get there via normal evolution. Note that many of these are in binaries. Scenario is that MSPs live/die in in binary as normal PSRs. The transfer mass, angular momentum. Reborn (recycled) as MSPs.

Lecture XVI Neutron Star Cooling

They start out very hot, $> 10^{11}$ K. How does that heat get transported away?

Mostly by neutrinos, especially at the start. T goes to 10^{9-10} K after a day. Neutrinos come from the interior, so they can be very efficient, but eventually they slow down and photons from the surface take over. This happens at 10^8 K, with $T_{\text{Eff}} = 10^6$ K (about what we observe for most NSs). Neutrinos will dominate for the first 10^{3-5} yr.

XVI.2 Neutrino Cooling

For $T < 10^9$ K, cools via neutrino emission from interior. They escape freely. Basic reactions are Urca:

$$n \to p + e^- + \bar{\nu}_e$$

$$p + e^- \to n + \nu_e$$

in equilibrium. Most things stay the same, but neutrinos carry away energy. This happens during core-collapse too, but degeneracy of the particles slows things down.

In equilibrium

$$\mu_n = \mu_p + \mu_e$$

And if degenerate,

$$E_F(n) = E_F(p) + E_F(e)$$

We can further use:

$$E_F(n) = m_n c^2 + \frac{p_F^2(n)}{2m_n}$$

and the same for p, and

$$E_F(e) = p_F(e)c$$

Since $p_F(e)$ depends only on n_e, and $p_F(p)$ depends on n_p, $p_F(e) = p_F(p)$ since $n_e = n_p$. So we have:

$$\frac{p_F(n)^2}{2m_n} = p_F(e)c\left(1 + \frac{p_F(p)}{2m_p c}\right) - Q$$

with $Q = 1.3\,\text{MeV}$. But Q will be $\ll$ the other terms, so:

$$E'_F(n) = \frac{p_F^2(n)}{2m_n} \approx p_F(e)c = E_F(e)$$

where E'_F is the Fermi energy minus the rest mass (the kinetic part). So then:

$$p_F(e) = p_F(p) \ll p_F(n)$$

$$E_F(p)' \ll E'_F(n)$$

During the decay, the only neutrons that can do it are at $E'_F(n) \pm k_B T$. And to make electrons and protons, we also need then to have energy $E'_F(n) \pm k_B T$, with the neutrino carrying off $\sim k_B T$.

But we have said that $p_F(p) \ll p_F(n)$. We cannot match the energy (which needs to be near E_F) and the momentum (which needs to be near $\sqrt{2mE_F}$) at the same time. So this cannot happen!

Need to modify this with a bystander particle:

$$n + N \to N + p + e^- + \bar{\nu}_e$$

$$p + e^- + N \to N + n + \nu_e$$

The N particles make sure that both momentum and energy are conserved.

Put all of the physics together, find $L_\nu \propto MT_9^8$. Very strong power of T, and the luminosity is $\propto M$. This is because all parts of the inside contribute: we are not limited by scattering and the need to get photons out from the surface.

Connect inside temperature and outside. The outside of the star will have a "normal" atmosphere with scale-height ($\sim k_B T/m_p g \approx 1\,\mathrm{cm}$). The inside will be isothermal at T_I.

$$T_{\text{Eff}} \approx 10^{-2} T \left(\frac{T}{10^9\,\mathrm{K}}\right)^{-1/8}$$

so roughly $10^{-2}T$. Like for WDs. And:

$$L_\gamma = 4\pi R^2 \sigma T_{\text{Eff}}^4 \approx 7 \times 10^{29}\mathrm{W} \left(\frac{T_{\text{Eff}}}{10^7\,\mathrm{K}}\right)^4$$

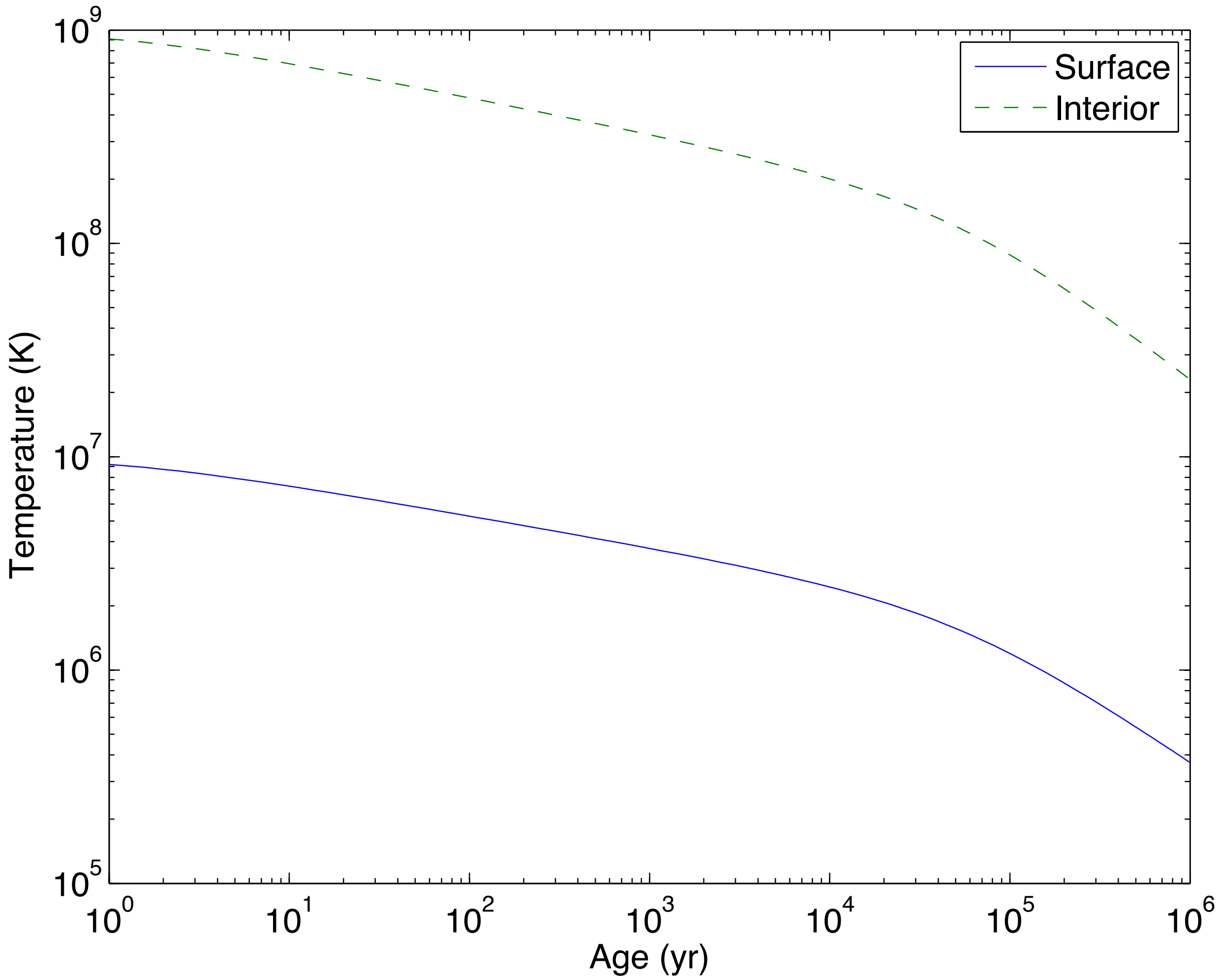
and

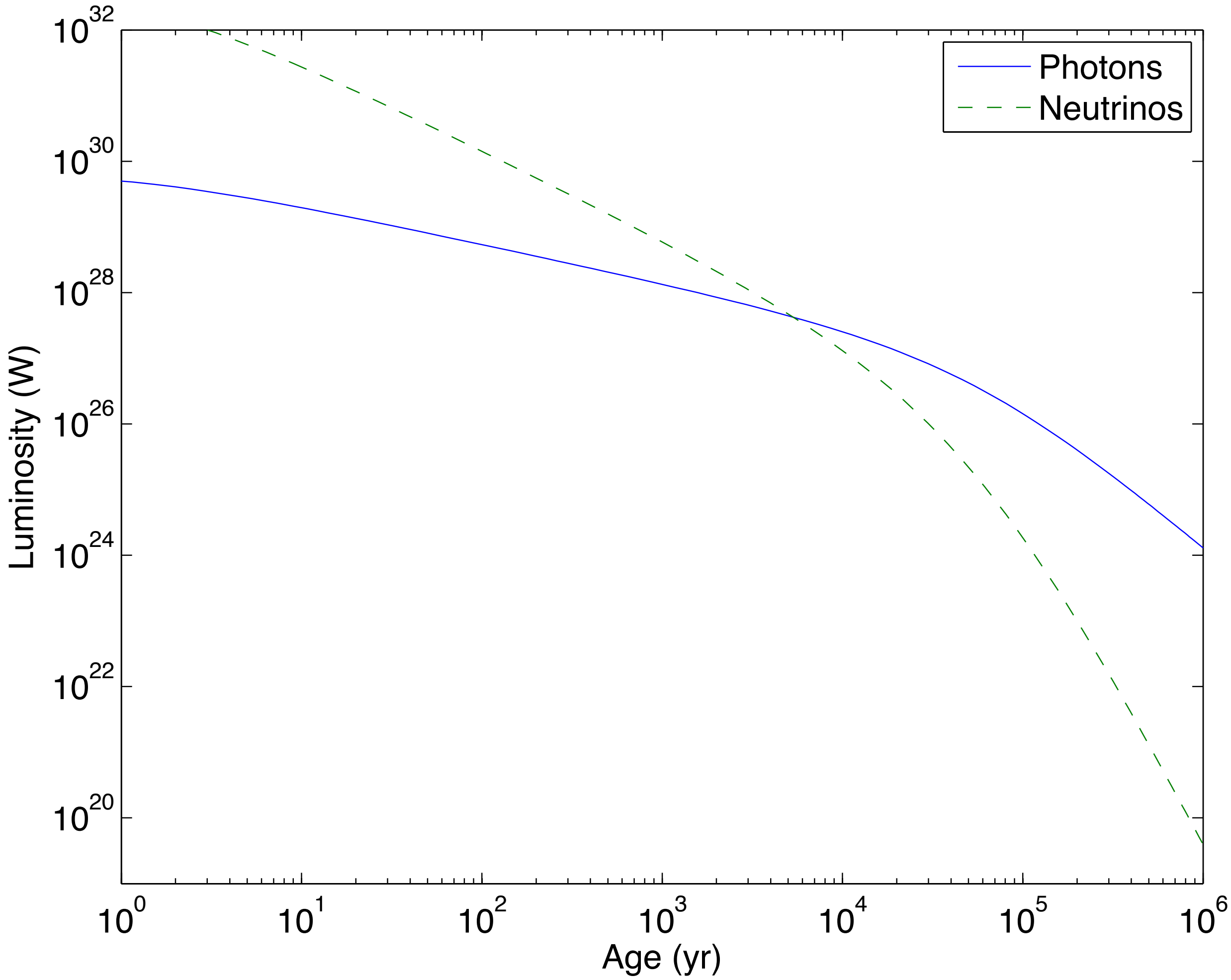
$$L_\nu \approx 5 \times 10^{32}\mathrm{W} \left(\frac{T}{10^9\,\mathrm{K}}\right)^8$$

So L_ν starts out much higher when T is high, but as T goes down it will go to lower than L_γ

Total thermal energy: roughly $\propto MT^2$. So

$$2TM\frac{dT}{dt} \propto -L_\gamma - L_\nu$$


Surface
Interior
Temperature (K)
Age (yr)
10^9
10^8
10^7
10^6
10^5
10^0
10^1
10^2
10^3
10^4
10^5
10^6



These temperatures on the surface are $\sim 10^6$ K, so X-rays. By measuring the cooling, can get at the physics inside. Probably have measured this for NS in Cas A, with age=300 yr. See change by 80,000 K in 9 years. This is faster than expected, so think that neutrons only recently became superfluid (when this happens, can get rid of a lot of energy).

Lecture XVII Magnetars!

Most neutron stars: energy is in the form of heat (when they are young) + rotation:

$$K = \frac{1}{2}I\Omega^2 = 8 \times 10^{43}\,\mathrm{J} = L_\odot \times 10^{10}\,\mathrm{yr}$$

(usually are brighter, so do not last as long). Pulsars are magnetized, but the magnet is only important because it gets the rotation energy out. Total luminosity $\dot{E} = I\Omega\dot{\Omega}$, but what we see $L_{\rm radio} \sim 10^{-4}\dot{E}$, $L_{\rm X-ray} \sim 10^{-3}\dot{E}$, $L_\gamma \sim 0.1\dot{E}$, synchrotron nebula $\sim \dot{E}$.

1979: burst of γ-rays from LMC. 10^{38} W ($10^{12}L_\odot$) spike, 3 min decay with 8 s pulsations. Now have seen several "giant flars" plus more numerous (but smaller) bursts (10^{34} W). Call these objects Soft Gamma-ray Repeaters. 1990s: X-ray pulsars with $P \sim 10$ s. Measure $\dot{P}$, find $\dot{E} \sim 10^{27}$ W. But see $L \sim 10^{29}$ W, so $L_X \gg \dot{E}$. How? Call these Anomalous X-ray Pulsars.

Chris Thompson & Rob Duncan: **magnetars** (also Paczynski). NS powered by decay of strong B, $B > 10^{10}$ T. Arguments:

- $B = 3.2 \times 10^{15}\sqrt{P\dot{P}}$ (like for pulsar) gives $> 10^{10}$ T
- Bursts are $\gg L_{\rm Edd}$. Remember, Eddington limit balances gravity with radiation pressure assuming pressure from σ_T:

  $$L_{\rm Edd} = \frac{4\pi G M m_p c}{\sigma_T} = 10^{31}\,\mathrm{W}\left(\frac{M}{M_\odot}\right)$$

  Does this mean $M \gg M_\odot$? No: we see pulsations, so the object must be small (few $R_\odot$). Instead, change σ. Strong B:

  $$\sigma = \sigma_T\left(\frac{B_{\rm QED}}{B}\right)^2$$

  with $B_{\rm QED} = 4 \times 10^9$ T, so $10^3 L_{\rm Edd}$ means $B > 30B_{\rm QED}$.
- Also, fireball to produce giant flares would blow itself apart. Strong B helps keep it tied to the NS surface so that it can rotate around.

XVII.2 Energetics

Sources are young (found in the Galactic plane, some in SNRs that only stay around for 10^4 yrs). Total energy $\sim L_X \times$age:

$$E \sim 10^{28}\,\mathrm{W}10^4\,\mathrm{yr} \sim 3 \times 10^{40}\,\mathrm{J} \sim \left(\frac{B^2}{2\mu_0}\right)\left(\frac{4\pi}{3}R_{\rm NS}^3\right)$$

So strong B can power what we see.

XVII.3 Conclusions

AXP, SGR are magnetars: NS with very strong B. Not powered by rotation but by decay of strong B (e.g., solar flares). Questions:

- How many?
- Why do they form this way?
- How are they different?
- How does the decay happen in detail?

Lecture XVIII Black Holes

Phillips 6.4

What happens if you add more mass to a NS? There's no other particle to support it. For $> 3\,M_\odot$ or so, cannot resist collapse. The gravitational redshift:

$$\frac{\Delta\lambda}{\lambda} = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2} - 1$$

goes to infinity. What about the escape speed? $v_{\text{esc}} = \sqrt{2GM/R}$ must be $< c$. This tells us that:

$$R > R_{\text{Schwarzschild}} \equiv \frac{2GM}{c^2}$$

for an object to not collapse. Or, we can write:

$$\frac{\Delta\lambda}{\lambda} = \left(1 - \frac{R_{\text{Sch}}}{R}\right)^{-1/2} - 1$$

so as R gets close to R_{Sch} the redshift approaches ∞. When $R = R_{\text{Sch}}$, we have a black hole (note that this derivation is not correct, but it gives the right answer). Once we squeeze any amount of mass into the corresponding R_{Sch}, we cannot get anything (light, information) out. For $1\,M_\odot$, $R_{\text{Sch}} = 3\,\text{km}$. Event horizon at R_{Sch}. As you fall toward a black hole, the gravity doesn't bother you. You don't feel anything as you pass the event horizon. But the tidal forces will get very strong and squeeze/stretch you.

We cannot detect black holes directly. Only indirectly. What we see is that gas (or other stuff) orbits a space that is really small. We can measure the redshift and blueshift on either side of the BH. If we can prove that it is orbiting in a space that is $< R_{\text{NS}}$ and has a mass that it $> M_{\text{NS}}$, we can say its a black hole.

However, this matter orbits only over a range of radii. For the Earth, as long as the orbit is at $> R_\oplus$, it's stable (ignore atmosphere). Same for WD and NS. But all BH orbits are not stable.

Consider orbits using the *Effective Potential*. Motion is a second-order DEQ. So we need 2 constants of the motion (initial conditions). Normally they would be $r(0)$ and $v(0)$ or something like that. But they don't have to be. In fact, it's often easier to write it in terms of conserved quantities rather than quantities at a specific time. So instead we will use $E/m = (1/2)v^2$ and $L/m = vr$.

A particle has an orbit defined by angular momentum. If this is 0, will drop straight in. Effective potential:

$$V(r) = U(r) + \frac{L^2}{2m^2r^2}$$

with L the angular momentum and $U(r)$ the potential. Use this in energy diagram to figure out orbits. For Newton,

$$U(r) = -\frac{GM}{r}$$

You move in following this until you hit the centrifugal barrier (L^2 piece), which makes you orbit back out. Stable orbits are in a well defined by the potential, can be elliptical or circular. For a given L, can plot the curve and see where you have circular orbits. These happen at:

$$r_C = \frac{l^2}{GM}$$

with $l = L/m$ for Newton. Can get this by differentiating:

$$\frac{dV}{dr} = \frac{GM}{r^2} - \frac{l^2}{r^3}$$

Set this equal to 0 to find the saddle point.

So for any value of l, have an r_c. Turning points are where $E = V$, defines bound vs. unbound. For $r = r_c$

$$V = E = -\frac{G^2M^2}{2l^2} = -\frac{G^2M^2}{2v^2r^2}$$

Or:

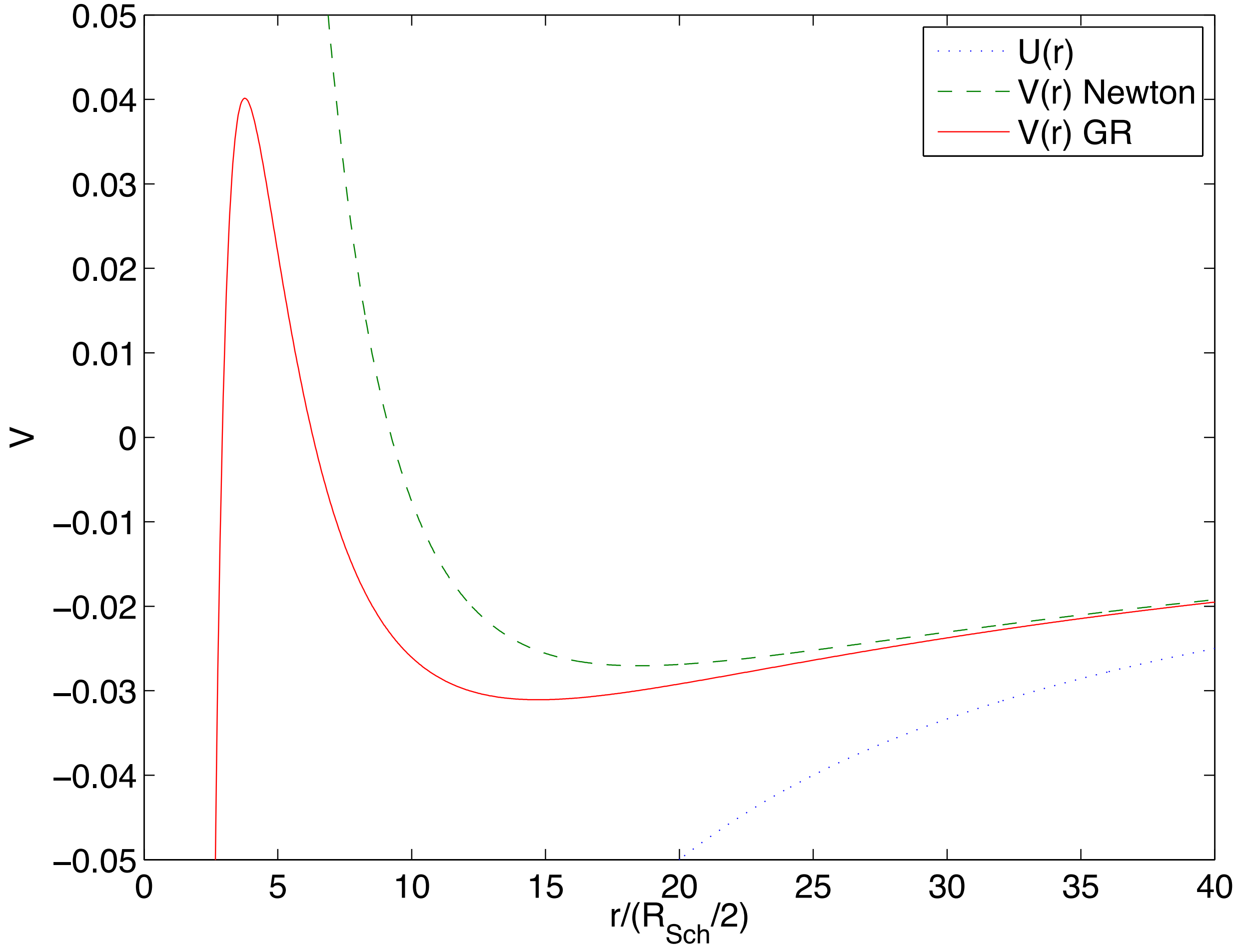
$$v = \sqrt{\frac{GM}{r}}$$

which is a result we could have gotten other ways.

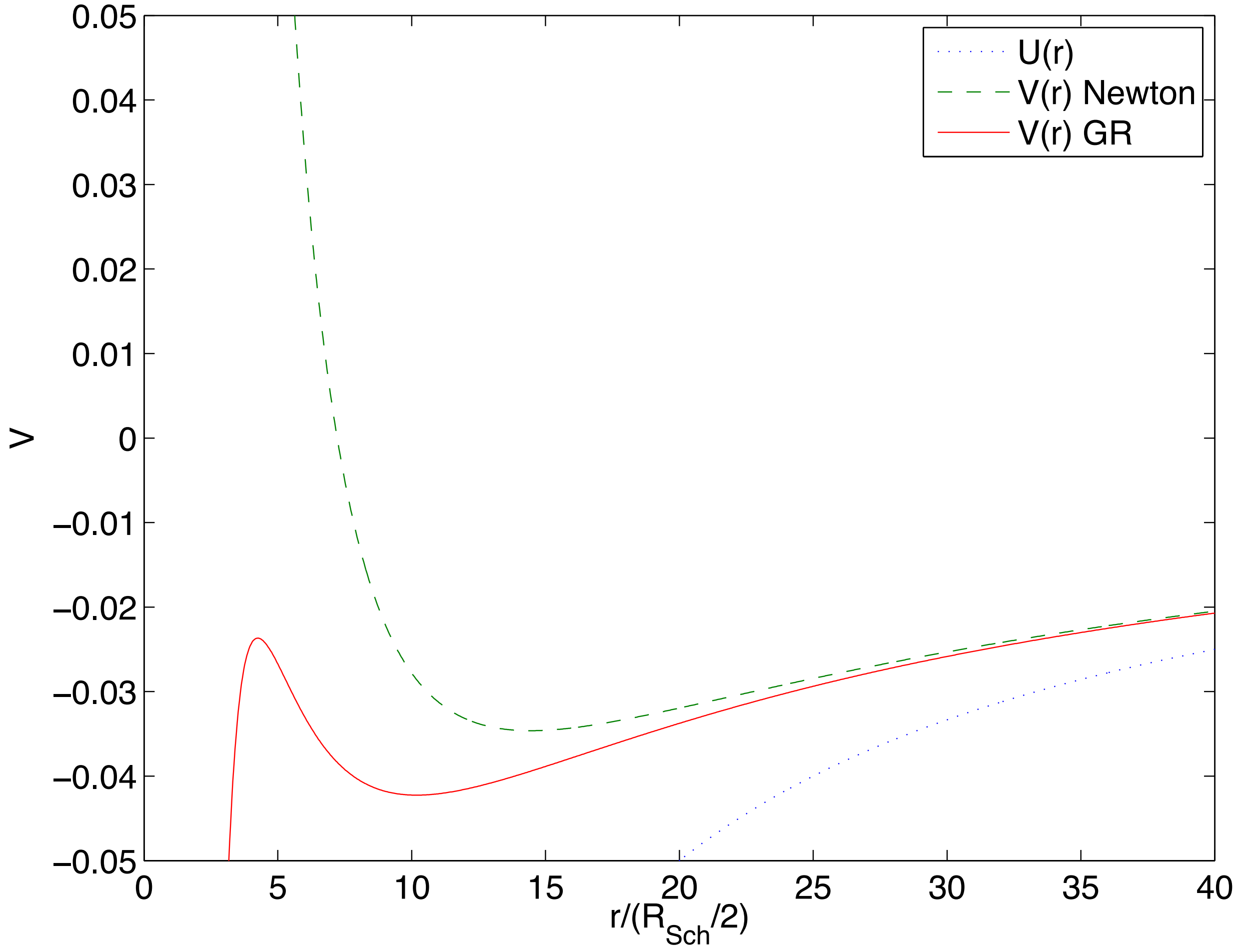
But this changes in GR. If the BH is not spinning (Schwarzschild solution), get:

$$V(r) = \left(\frac{-GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}\right)$$

This puts another negative piece on, so the barrier has a finite height. You can come in with some energy (depending on l) and go over the barrier, and then you will fall all the way in to the center.


U(r)
V(r) Newton
V(r) GR
V
r/(R_Sch/2)
0.05
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
−0.05
0
5
10
15
20
25
30
35
40

For $l = 4.3GM/c$.



For $l = 3.8GM/c$.

So potentials have no stable circular orbits (valleys), if l is low enough. Look for what the smallest possible stable circular orbit is. Height of the barrier from differentiating:

$$\frac{dV}{dr} = \frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{c^2r^4}$$

Set this equal to 0, find:

$$r_c = l\frac{cl \pm \sqrt{c^2l^2 - 12G^2M^2}}{2GMc}$$

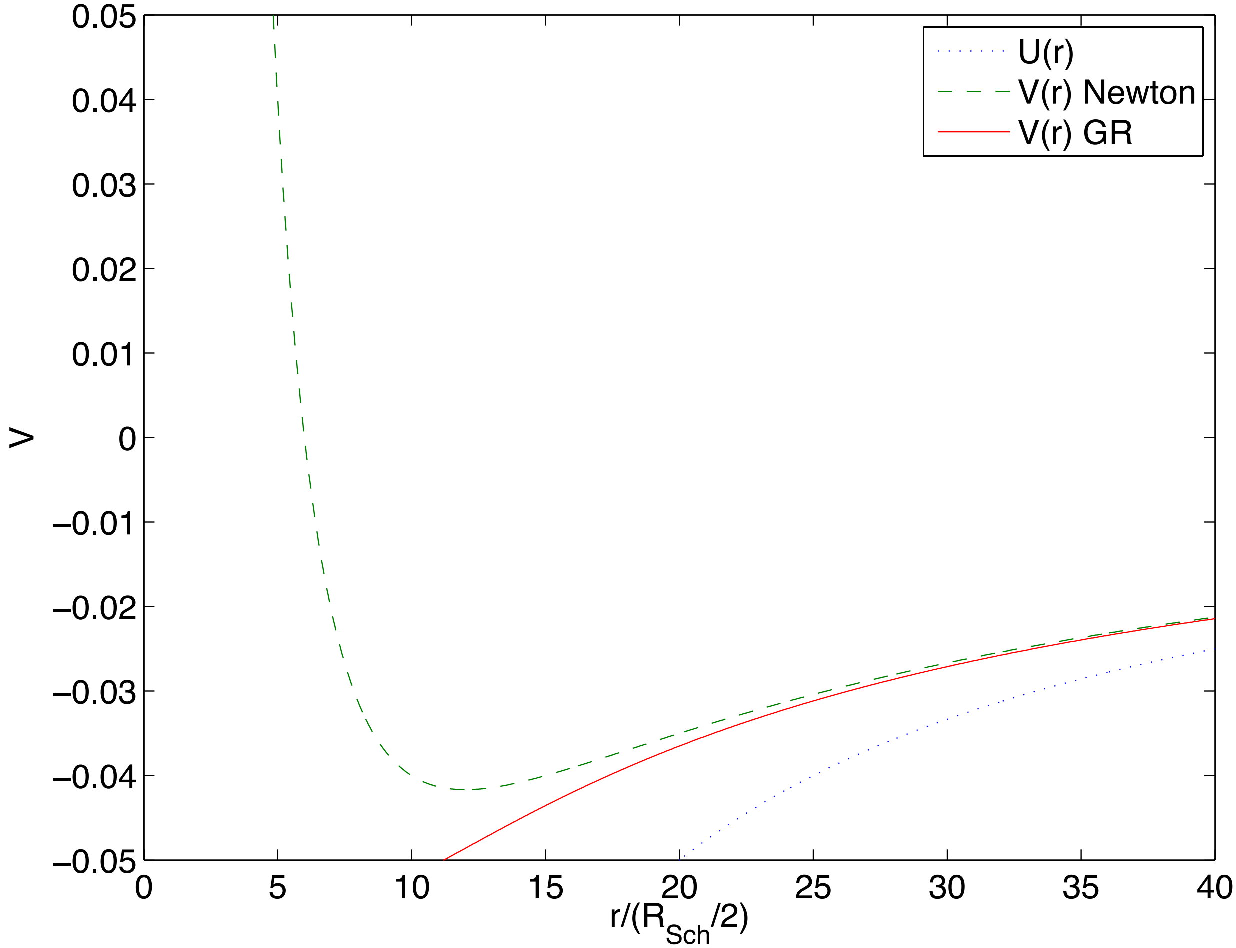
One of these is stable (valley), the other is unstable (peak), and the unstable one is the inner one. They will be the same when the discriminant vanishes, or $l = \sqrt{12}GM/c$. At this point,

$$r_c = 6GM/c^2 = 3R_{\rm Sch}$$

This is known as Innermost Stable Circular Orbit (ISCO): anything that gets inside here will not be on a circular orbit. And eventually, it will likely fall into the BH.

Can also see that if E is too high, will skip right over the barrier and fall into the BH. This does not happen in Newton: there, as long as $l > 0$, the orbit will always stay at $r > 0$. For a fixed l, this will happen if E is high. What does that mean? $E/m = v^2/2$ and $l = vr$, so $v = l/r$ and

$E/m = l^2/2r^2$. So having a high E at a fixed l means that r is small, or that the orbit would come close to the center. So this makes sense.



For $l = \sqrt{12}GM/c$.

This matter orbiting around and falling into a BH is what allows us to identify them. See a disk (outside ISCO) and a jet. BH can be big (center of galaxy, supermassive, $10^{6-9}\,M_\odot$) or small (stellar, microquasar). As the matter falls in, liberates some fraction of the gravitational binding energy. If the matter comes in at a rate $\dot{M}$, goes from $r = \infty$ to $r = R$ (if has a surface) or r_c (if BH). Then energy liberated is:

$$L \sim \frac{GM}{R}\dot{M}$$

Maximum this can be is Eddington luminosity: at that point, the pressure from the light would be enough to blow away any more material from falling in.

Eventually, GR will win, and even orbits that seem stable will not be. Some energy is slowly lost to gravitational radiation. So things spiral in together. For instance, NS+NS may merge, forming GRB.

The ISCO changes if the BH is spinning. If matter orbits around the same way as the spin, then can come in a bit closer. If the other way, needs to stay out farther. So found BHs with matter orbiting within $6GM/c^2$, can say that the BH is spinning.

XVIII.2 Black Hole Scalings

Common when working on relativity to use relativistic units. Instead of SI/mks or cgs, use units that emphasize the basic scalings.

There are three meaningful numbers for units: length, time, mass. Geometric units have $G = c = 1$. So we can re-write the Schwarzschild radius:

$$R_{\rm Schwarzschild} \equiv \frac{2GM}{c^2} \rightarrow 2M$$

This leads to units where M is mass and length and time. Everything ends up as length.

$c = 1$ connects length and time. So $R = x$ and $t = x/c$ in normal units become $R = t = x$ (in meters).

Add in G, and you can connect mass too. G/c^2 is the conversion factor, and with that we get masses in terms of lengths. So $1\,M_\odot = 2\times 10^{30}$ kg becomes $GM_\odot/c^2 = 1.5\,$km. Then $R_{\rm Sch} = 2M = 3\,$km is easy to remember.

We still need constants h (or $\hbar$), k_B, and ϵ_0. We could set one more of these to 1, and it is common to set $\hbar = 1$, but we won't worry about that.

This makes the math a bit easier to do sometimes (removes all of the powers of G, c, etc.). But it makes it harder to check units, and you often need to put the units/constants back when trying to measure something physical.

Example: effective potential. It was:

$$V(r) = \left(\frac{-GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}\right)$$

If we write it in geometrized units, we get:

$$V(r) = \left(\frac{-M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\right)$$

For a solar-mass object, $M = 1.5\,$km. r will also be in units of length, and in fact it is useful to write it in terms of M:

$$V(x) = \left(\frac{-1}{x} + \frac{l^2}{2M^2x^2} - \frac{l^2}{M^2x^3}\right)$$

with $x = r/M$. Angular momentum L has units of mass$\times$velocity$\times$length, or mass$\times$length^2/time. So in geometrized units we just end up with length^2, and to get there we need to multiply by G/c^3. But we are actually dealing with $l = L/m$, which has units of length^2/time or length in geometrized units. And as you can see, it is more useful to look at the problem in terms of $y = l/M$. Then we have:

$$V(x) = \left(\frac{-1}{x} + \frac{y^2}{2x^2} - \frac{y^2}{x^3}\right)$$

which is a lot cleaner. We could do the same work we did before to find:

$$x_c = y\frac{y \pm \sqrt{y^2 - 12}}{2}$$

for the circular orbits, and that the ISCO is for $y = \sqrt{12}$ with $x_c = 6$.

Lecture XIX ISM & Star Formation

XIX.2 Interstellar Medium

What is between the star? What goes in to making the stars, and what happens after supernova?

- gas: 99%
- dust: 1%

Like stars, this is mostly H and He.

[Hydrogen is H. When it is atomic, it is H I. When it is an ion, it is H II. When it is a molecule, it is H_2 (confusing).]

Dust causes light to be dimmer and redder. Not a lot of mass, but important effects: we cannot see a large portion of the Galaxy at optical wavelengths. Need to go to infrared.

XIX.2.1 Gas

This is most of the mass of the ISM, but still only $\sim 10\%$ of the mass of stars. It can have multiple phases:

- warm vs. cold
- dense vs. diffuse
- atoms vs. molecules

Average is $n \sim 10^6\,\mathrm{m}^{-3}$ (best vacuum on Earth is $10^9\,\mathrm{m}^{-3}$). Numbers here are for Milky Way:

Component	Volume	T (K)	n (m^{-3})	State	See via
molecular clouds	$< 1\%$	10–20	10^{8-12}	H_2	molec. lines, em. and abs.
CNM	1–5%	70	3×10^7	H	H I 21 cm abs
WNM	10–20%	10^4	10^6	H	H I 21 cm em
WIM	20–50%	10^4	10^6	H II	Hα
H II regions	$< 1\%$	10^4	10^{8-10}	H II	Hα
HIM	30–70%	10^{6-7}	10^{2-4}	H II	X-ray

The different phases are roughly at pressure equilibrium: $P = nk_BT$ is roughly constant.

Molecular clouds are small and dense: that's where stars are born.

H II regions surround hot stars (or hot WDs).

The rest takes up the space in between.

XIX.3 Star Formation

Star formation is not well understood, although we have learned a lot over the last 10 years from infrared and longer-wavelength observations. That's because star-formation takes place where the gas is dense and cold→warm (but not hot), so it radiates primarily in the infrared.

Basic facts:

- In our Galaxy, the overall steady-state rate of star formation (SFR) is $\sim 1\,M_\odot/\mathrm{yr}$. As this goes on, stars of a range of masses are formed.
- Initial Mass Function (IMF): the distribution of stellar masses when they are formed. Salpeter (1955) was the first to derive an empirical IMF for stars near the Sun. He took the observed luminosity function (how many stars of a given luminosity) with the mass-luminosity relation to get the mass function $\Phi(M)$. With this, the number of stars with masses between M and $M + dM$ is $\Phi(M)dM$.

  Call the IMF $\Psi(M)$. How does this relate to $\Phi(M)$ (the distributio of observed masses)? For stars that live longer than the Galaxy ($\tau(M) > \tau_{\rm MW}$ for $M < M_0$), all of the stars will still be present, so $\Phi(M) = \Psi(M)$. But for stars that do not live that long (higher mass stars), the observed number of them will be dimished. We can divide by that ratio $\tau(M)/\tau_{\rm MW}$ to figure out what factor we need to replace those stars in our counting. Or:

  $$\Psi(M) = \begin{cases} \Phi(M) & \tau(M) > \tau_{\rm MW} \\ \Phi(M)\left(\frac{\tau_{\rm MW}}{\tau(M)}\right) & \tau(M) < \tau_{\rm MW} \end{cases}$$

  If we just counted naively, we would underestimate the number of massive stars. Salpeter found:

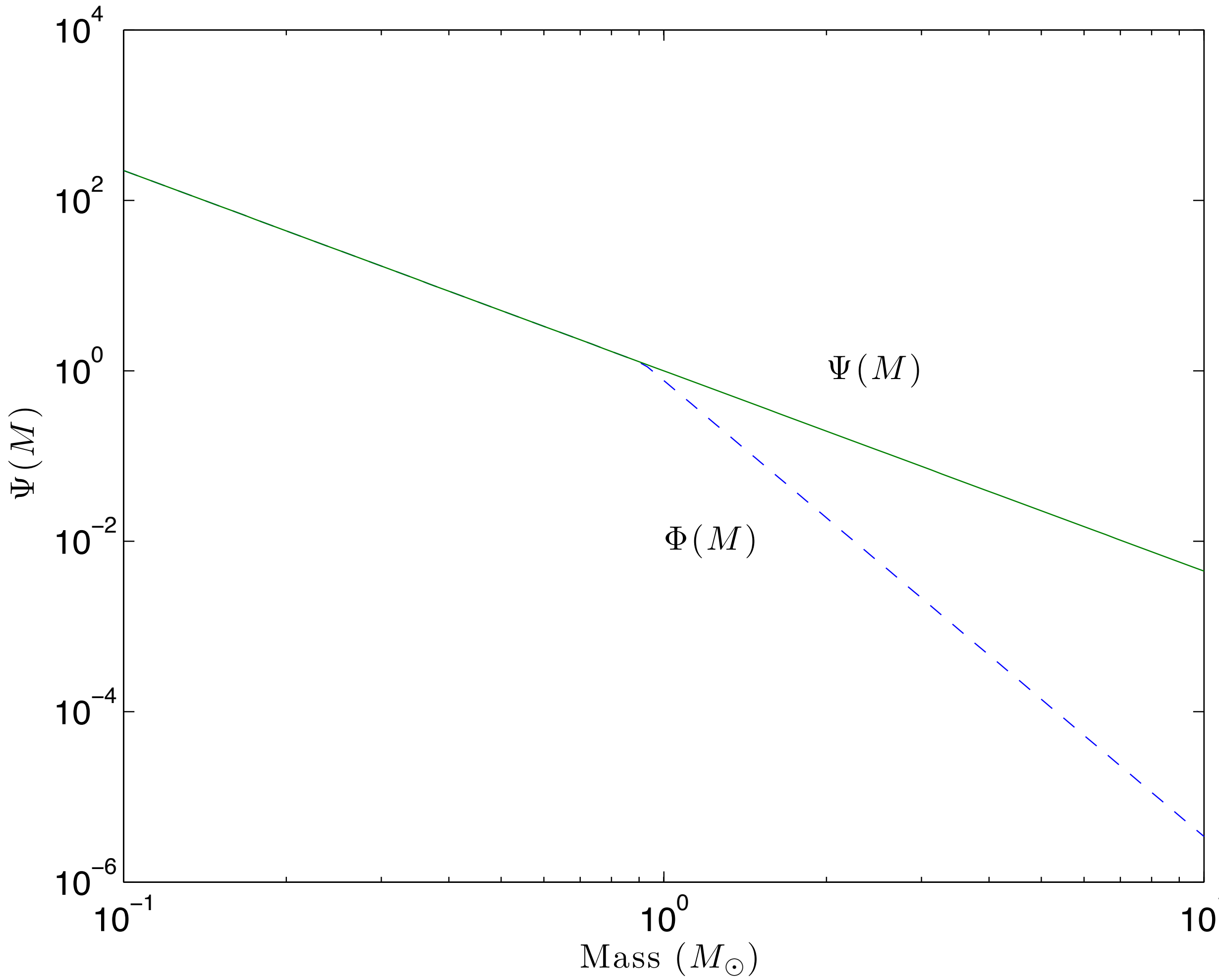
  $$\Psi(M) \propto M^{-2.35}$$

  for $0.4\,M_\odot$ to $10\,M_\odot$. Modern studies suggest that the IMF is flatter for lower masses ($< 0.5\,M_\odot$ or so) and steeper for higher masses, but this is an ongoing area of research. But this says that many more low-mass stars are formed compared to high-mass stars.

  Open questions:

  – Is the IMF the same from place-to-place (within a galaxy or between galaxies)?
  – Can we predict the IMF from the fundamental theory of star formation (not yet)?

- Sites of SF in our Galaxy: cool molecular clouds. These have $T \sim 10\,\mathrm{K}$, $n \sim 10^9\,\mathrm{m}^{-3}$, and $R \sim 5\,\mathrm{pc}$, so $M \sim 10^3\,M_\odot$. Giant molecular clouds have $M \sim 10^5\,M_\odot$ and $R \sim 20\,\mathrm{pc}$.
  In contrast, the density of star is $\rho \sim 10^3\,\mathrm{kg\,m^{-3}}$ or $n \sim 10^{30}\,\mathrm{m}^{-3}$. So significant compression must occur between the ISM, the molecular clouds, and stars. Gravity wins, but it has to overcome gas pressure, rotation, magnetic fields. This is not easy to do from first principles.



XIX.4 Jeans Instability

To collapse into a star, gravity must be stronger than pressure (kinetic energy).

$$U = -f\frac{GM^2}{R}$$

(f depends on density distribution, $f \sim 1$).

$$K = \frac{3}{2}Nk_BT$$

Need $|U| > K$ for collapse. Can write this as:

$$M > M_J = \frac{3k_BT}{2G\bar{m}}R$$

where $\bar{m}$ is average mass of particle ($M = N\bar{m}$). Or

$$\rho > \rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_BT}{2G\bar{m}}\right)^3$$

or

$$M > M_J = \left(\frac{3}{4\pi\rho}\right)^{1/2}\left(\frac{3k_BT}{2G\bar{m}}\right)^{3/2}$$

These are the **Jeans mass and density**. M_J is $\propto \rho^{-1/2}T^{3/2}$: if T is higher more pressure support, but if denser than collapse is easier.

When collapse starts, the cloud is near equilibrium. But as the collapse goes on, the gravitational energy $|U|$ becomes bigger and bigger with respect to K, so the collapse accelerates. In equilibrium, Virial theorem says:

$$2K + U = 0$$

But when collapsing:

$$2K < -U$$

As the collapse starts, the cloud is transparent to radiation: any conversion of gravitational energy is radiated away. So if any part is hotter it will radiate and heat up other parts. So during the initial stages of collapse T remains largely constant (isothermal collapse). This means that K stays constant. But $U \sim -GM/R$ is increasing in magnitude as R decreases. Eventually, the collapse would accelerate to near free-fall, with:

$$\tau_{\rm ff} \sim \frac{1}{\sqrt{G\rho}}$$

which is $\sim 10^3$ yr for the dense core of a GMC.

XIX.4.1 Fragmentation

Conditions necessary to collapse: $T = 20\,\mathrm{K}$, $M = 10^3 M_\odot$, needs $\rho = 10^{-22}\,\mathrm{kg/m^3}$ ($n = 10^5\,\mathrm{m^{-3}}$) to collapse (not too bad). But for $1M_\odot$ density needs to be 10^6 times higher. That is harder.

So want a big cloud to collapse. But does a big cloud make a big star? Generally it breaks up along the way (*fragmentation*).

Imagine a collapsing cloud with $M > M_J$. As it collapses, ρ increases, so the local M_J decreases ($\propto \rho^{-1/2}$ if T is constant). So the whole thing cannot collapse as a whole, but breaks up.

This process will stop when the gas becomes opaque, so that the local cooling time is $>$ the free-fall time. What is the fragment mass M_F? (Rees 1976). Consider fragment with:

$$M \sim M_J \sim \left(\frac{k_BT}{\bar{m}G\rho}\right)^{3/2}\rho$$

Collapse is with $\tau_{\rm ff} \sim 1/\sqrt{G\rho}$. In contrast, cooling time is:

$$\tau_{\rm cool} \sim \frac{Mk_BT}{\bar{m}L} \sim \frac{Mk_BT}{\bar{m}4\pi R^2\sigma T^4}$$

along the lines of the Kelvin-Helmholtz time. Set these equal to find when fragmentation stops:

$$\frac{1}{\sqrt{G\rho}} < \frac{Mk_BT}{\bar{m}4\pi R^2\sigma T^4} = \frac{Mk_BT}{\bar{m}4\pi\sigma T^4(3M/4\pi\rho)^{2/3}}$$

But we also have $M \sim M_J$, so can eliminate ρ to find

$$M_F \sim \left(\frac{1}{4\pi}\left(\frac{4\pi}{3}\right)^{2/3}\right)^{1/2}\left(\frac{k_B}{\bar{m}}\right)^{9/4}\frac{T^{1/4}}{G^{3/2}\sigma^{1/2}}$$

Which simplifies to $\sim 0.002\,M_\odot$. So why aren't stars that small? Various things likely stop fragmentation first.

XIX.4.2 Angular Momentum Problem

What about rotation? If a blob of gas is on the equator of a rotating cloud, it will have:

$$J = \Omega R^2 = \text{const}$$

(angular momentum per unit of mass, with frequency Ω). Angular momentum is conserved for the blob. Gravity is a force $\sim GM/R^2$, while centrifugal forces are $\sim J^2/R^3$. So if J is not too small, as R decreases eventually centrifugal forces will be important. What is J?

Start with a $1\,M_\odot$ cloud. $n_0 \sim 10^7\,\mathrm{m}^{-3}$, so $\rho_0 \sim 10^{-20}\,\mathrm{kg\,m}^{-3}$. From this $R_0 \sim (3M/4\pi\rho_0)^{1/3} \sim 3\times10^{16}\,\mathrm{m} \sim 1\,\mathrm{pc}$.

What is Ω? Galactic rotation has a period of roughly $P_G \sim 2\times10^8$ yr. The outer part of the blob will be going around the galaxy at a different speed than the inner part, which will lead to $\Omega > 0$. We can expect:

$$\Omega_0 \sim \frac{2\pi}{P_G}\beta \sim 10^{-15}\beta\,\mathrm{s}^{-1}$$

where $\beta < 1$ is a constant. From this we get $J_0 \sim R_0^2\Omega_0 \sim 10^{18}\beta\,\mathrm{m}^2\,\mathrm{s}^{-1}$. When will this be important? When:

$$\frac{GM}{R^2} \sim \frac{J^2}{R^3}$$

or when $R \sim J^2/GM \sim 10^{16}$ m. So already at 1 pc it is important, and $R_\odot$ is $\ll$ this.

How do we get rid of J?

- Break into pieces, and then the internal rotation (bad) is transferred into orbital motion (OK). This makes binary stars or groups of stars, which we see.
- Outflows from the disk. Just like in quasars, the disk of matter collapsing in to form the star can launch jets, with help from the magnetic field. This can transport J to ∞, and allow the cloud to collapse.

XIX.4.3 Magnetic Fields

These can also be problematic.

$$P_B \sim \frac{B^2}{2\mu_0}$$

(magnetic pressure = energy density). With interstellar $B \sim 10^{-10}$ T, need to redo collapse criteria including this effect. Complicated, because direction-dependent. And flux-freezing means $BR^2 \sim$ constant.

Lecture XX H II Regions

Hydrogen is H. When it is atomic, it is H I. When it is an ion, it is H II. When it is a molecule, it is H_2 (confusing).

Consider a newly formed star. it will be embedded in a dense cloud of H gas. The star emits photons which are energetic enough to ionize H, making H II. How big a bubble of ionized gas will there be? We call this an H II region.

The star is hot enough ($> 10^4$ K) that photons have $h\nu > 13.6\,\mathrm{eV}$:

$$\mathrm{H\,I} + \gamma \to p + e^-$$

Take N_* is the number of photons per second coming from the star with $E > 13.6\,\mathrm{eV}$ (beyond the Lyman limit, or $\lambda < 912\,\mathrm{Å}$). Assume each photon hits one atom and is absorbed making an ion.

For every ion (p), there is a chance that it will encounter an e^- and recombine:

$$p + e^- \to \mathrm{H\,I}$$

In equilibrium, this will balance ionization. But no Saha (since this is driven by the star). Instead:

$$\mathcal{R} = \frac{\text{number of recombinations}}{\text{volume} \times \text{time}}$$

is the recombination rate, determined by the gas properties. This will balance ionization:

$$\mathcal{R}V = N_*$$

or

$$\mathcal{R}\frac{4\pi r^3}{3} = N_*$$

What is $\mathcal{R}$? It depends on the rate at which e^- hits p. So it depends on the product of the densities, $\mathcal{R} \propto n_e n_p = \alpha n_e^2$. Then

$$r = \left(\frac{3N_*}{4\pi\alpha n_e^2}\right)^{1/3}$$

defines the size of the Strömgren sphere. For example, $\alpha = \alpha(T) = 3 \times 10^{-19}\,\mathrm{m^3\,s^{-1}}$ at 10^4 K, $\alpha(T) \propto T^{-1/2}$ is the recombination coefficient, $n_e \sim 10^8\,\mathrm{m^{-3}}$, $N_*(\mathrm{O5}) = 3 \times 10^{49}\,\mathrm{s^{-1}}$ gives r of a few pc.

Hotter or bigger star (higher N_*) gives a bigger region. Denser gas makes it smaller.

If we have multiple stars we can have a bigger region. Or we can have a hot WD and a smaller region (planetary nebula). In general, the brigher H II regions tell us where massive stars are (in this galaxy and others).

Lecture XXI Dust

Obvious as dark patches in the sky ("holes" in the distribution of stars). These are not actually missing stars, but the light is blocked by dust.

Originally worked out by Trumpler (1930): star cluster has a bunch of stars at the same distance. These should line up on the main sequence. So we can move them up & down (guessing distances) until they line up with the model.

Trumper said: d =distance, D =diamter. So the angular size is

$$\theta = \frac{D}{d}$$

The angular area will be:

$$\theta^2 = \frac{D^2}{d^2} \propto \frac{1}{d^2}$$

if most clusters are similar.

At the same time, if the luminosity of a star is L, its apparent brightness (flux) is:

$$F = \frac{L}{4\pi d^2} \propto \frac{1}{d^2}$$

if most clusters are similar.

So if L and D are typical values that don't depend on each other, we should see:

$$\theta^2 \propto F \propto \frac{1}{d^2}$$

We do not know d, but we can measure θ and F directly. Plot these:

- for high θ^2 these should have high F, and will be nearby.
- for low θ^2 these will have low F and will be more distant. But Trumpler found that the fluxes were less than that predicted by the model.

There was scatter (since not all clusters are the same) but also a systematic departure.

Further away were dimmer than expected, and these were also redder. Both of which are caused by dust.

Light from the star comes into the cloud. Red light is preferentially transmitted, bluer light is reflected and scattered into all directions (like a sunset). Also, absorbed (converted into heat, which makes dust warm and glow in IR).

In air, color change mostly from Rayleigh scattering:

$$\propto \frac{1}{\lambda^4}$$

since the scattering objects (molecules) are $\ll \lambda$. With stars, though, the scattering is:

$$\propto \frac{1}{\lambda}$$

so the scattering cannot be the same stuff. We can infer that it must be $\sim \lambda$ in size. We now know that dust is little balls of C and Si atoms.

In the optical dust makes stars appear dimmer and redder (**show on HR diagram**). In the infrared, though, we can see dust directly and use that to map the heating, or see through dust to map the whole Galaxy.

Lecture XXII Supernova Explosions

Collapse of iron core of massive star.

10^{46} J released, which is mostly the gravitational energy before collapse.

Of this:

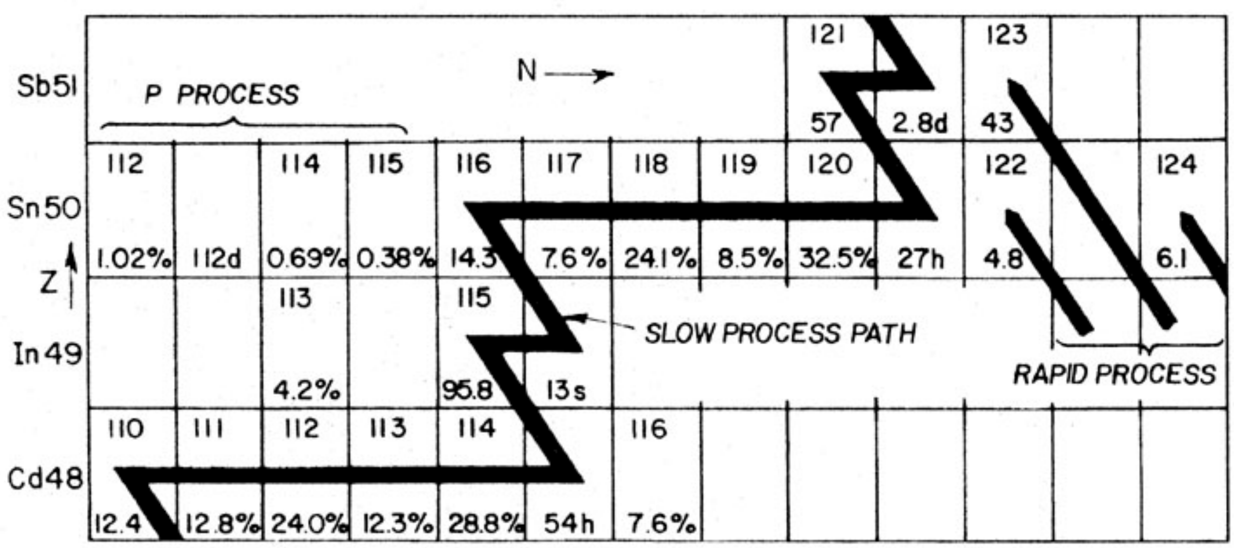
- 1% goes into the kinetic energy of the explosion (10^{44} J is 10^{51} erg, so this is known as 1 foe).
- 0.01% goes into photons
- 99% goes into neutrinos

XXII.1.4 Chemistry

Sun produces mostly He from fusion. Massive stars can produce up to Fe, but still limits (i.e., little Li produced). How to make the rest? SNe.

Most of this happens through the *r-process* (rapid). Heavy elements + many neutrons → very heavy, unstable nuclei. Then decay to something stable (but still heavy).

The *s-process* (slow) can also occur, but that is mostly in post-main sequence evolution, where there is repeated n capture then α or β decay.



XXII.1.5 Qualitative

Flings fast, hot gas outward. At the edge of the shock the gas is $\sim 10^7$ K, generating X-rays. It runs into the ISM with $10^6\,\mathrm{m}^{-3}$, 10^4 K.

Blast wave solution:

1. free expansion (few $\times 10^{3-4}\,\mathrm{km\,s^{-1}}$, supersonic)
2. Sedov (sweeps up mass comparable to ejecta mass); conserve energy; Derive self-similar solution
3. radiative/snow-plow: conserve momentum

Show sound-speed of gas, compare to blast-wave speed, estimate time for Sedov solution.

$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\gamma P}{\rho} = \frac{\gamma k_B T}{\bar{m}}$$

XXII.1.6 Shock Waves

Sketch what happens to a wave disturbance when it starts to go non-linear.

For example, consider the propagation of a finite amplitude sound wave. The speed of propagation c_s is higher at higher temperature ($c_s \propto \sqrt{T}$). A wave has parts with higher and lower pressure, and through adiabatic equation of state ($P\rho^\gamma$ =constant) T and ρ are also different when P is higher ($T \propto \rho^{\gamma-1}$), so the different parts also have different speeds. Eventually the crest of the wave (higher ρ and P) gradually overtakes the trough. When faster moving gas overtakes slower moving gas, we get a discontinuous change of density and velocity, a shock.

Hydrodynamics: mass conservation + momentum conservation + energy conservation.

So far we have dealt with static (non-moving) fluids. What happens if it moves? Look in 1D. Imagine box with size Δx. Stuff comes in, stuff changes, stuff goes out, all in x direction. Stuff can change with position or time:

$$\mathrm{stuff} = \mathrm{stuff}(x,t)$$

Stuff flowing in/out is a *flux* of stuff (not just energy). Can write as:

$$\mathrm{flux\,of\,stuff} = \mathrm{stuff} \times v$$

Comes in with v_1, comes out with v_2, and it can change in the box:

$$\frac{\partial}{\partial t}\mathrm{stuff} = \frac{\mathrm{stuff}\,v_1 - \mathrm{stuff}\,v_2}{\Delta x}$$

But:

$$\mathrm{stuff}\,v_1 = \mathrm{stuff}\,v(x)$$

and:

$$\mathrm{stuff}\,v_2 = \mathrm{stuff}\,v(x+\Delta x)$$

So we can take the limit as $\Delta x \to 0$ and get:

$$\frac{\partial}{\partial t}\mathrm{stuff} = -\frac{\partial}{\partial x}(\mathrm{stuff}\,v)$$

XXII.1.6.1 Density

If stuff=density (ρ), then:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

We can't make more stuff (i.e., mass) in the box. So this is basically conservation of mass.

XXII.1.6.2 Momentum

If stuff=momentum (mv), it's actually again easier to divide by volume of box to get momentum density ρv.

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2)}{\partial x} = 0$$

We can expand the derivatives:

$$v\frac{\partial}{\partial t}\rho + \rho\frac{\partial v}{\partial t} + v\frac{\partial}{\partial x}\rho v + \rho v\frac{\partial}{\partial x}v = 0$$

But the 1st and 3rd terms here are v times the mass conservation law from before, so their sum is 0. Therefore we have:

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = 0$$

This is most of the way there, but unless make we can make new momentum. How? Change in momentum per time is known as force. If there is an external force (like pressure) that can change the momentum. If the pressue is the same everywhere it won't do anything, but if it pushes more on one side vs. the other we will have a net force. So the final equation is:

$$\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

XXII.1.6.3 Energy

If stuff=energy:

$$\frac{\partial(E)}{\partial t} + \frac{\partial(vE)}{\partial x} = 0$$

But pressure is energy per volume, so we have to include it too:

$$\frac{\partial(E)}{\partial t} + \frac{\partial}{\partial x}(v(E+P)) = 0$$

With those three equations we can study shock waves.

XXII.1.6.4 Shocks

Imagine we have a piston pushing some gas down a tube in 1D. Upstream the material is undisturbed. Then a shock wave comes through and changes it. Downstream it has changed. Upstream parameters ρ_1 etc. The piston moves up the tube with speed v_p which is the speed of the post-shock material. The disturbance will propagate with speed v_s which is faster.

We can also think of this as fixed in the frame of the shock, with the gas moving. So the upstream material will flow *into* the shock with speed v_1, and $v_s = 0$. In that frame nothing changes with time, only position. So $\partial/\partial t = 0$. So we have:

$$\frac{\partial}{\partial x}\rho v = 0$$

$$\rho v\frac{\partial}{\partial x}v = -\frac{\partial P}{\partial x}$$

and

$$\frac{\partial}{\partial x}(E+P)v = 0$$

We can rewrite the second equation as:

$$\frac{\partial}{\partial x}\rho v^2 + \frac{\partial P}{\partial x} = 0$$

The shock itself is infinitely thin, and quantities will change very quickly across there. We don't look at the differentials. Instead we integrate over the shock and look at the two sides, upstream (1) vs. downstream (2). We have $\rho_2 > \rho_1$, $v_2 < v_1$, $T_2 > T_1$, $P_2 > P_1$. **sketch**.

$$v_1\rho_1 = v_2\rho_2$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$$

$$(E_1 + P_1)v_1 = (E_2 + P_2)v_2$$

These are known as (Rankine-Hugoniot) *Jump Conditions*.

However, writing E like this isn't so useful. Instead we use:

$$E = \frac{1}{2}\rho v^2 + \rho\epsilon$$

where ϵ is internal energy per mass (say from internal degrees of freedom). We can relate that to the pressure (which is energy per volume) via:

$$P = \rho(\gamma-1)\epsilon$$

with γ the adiabatic index of the gas ($5/3$ for monatomic). So:

$$(E+P) = \frac{1}{2}\rho v^2 + \frac{P}{(\gamma-1)} + P = \frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma-1}P$$

So:

$$v(E+P) = \rho v \left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho}\right)$$

and therefore the energy balance condition can be done as:

$$\rho_1 v_1 \left(\frac{1}{2}v_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}\right) = \rho_2 v_2 \left(\frac{1}{2}v_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2}\right)$$

But $\rho_1 v_1 = \rho_2 v_2$, so we can divide that part out:

$$v_1^2 + 2\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = v_2^2 + 2\frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2}$$

In a shock, a lot of the initial ordered kinetic energy from the supersonic motion gets converted into random motion (heat), making the downstream material hot. It converts supersonic to subsonic motion.

Define Mach number in terms of upstream sound speed:

$$\mathcal{M} = \frac{v_1}{a_1}$$

with a_1 is the sound speed, $a_1^2 = \gamma P/\rho$.

In terms of Mach number, jump conditions are now:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

$$\frac{P_2}{P_1} = \frac{\rho_2 k_B T_2/\bar{m}_2}{\rho_1 k_B T_1/\bar{m}_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}$$

which imply:

$$\frac{T_2}{T_1} = \frac{[(\gamma-1)\mathcal{M}_1^2 + 2][2\gamma\mathcal{M}_1^2 - (\gamma-1)]}{(\gamma+1)^2\mathcal{M}_1^2}$$

For strong shocks, $\mathcal{M} \gg 1$:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} \approx \frac{\gamma+1}{\gamma-1} = 4$$

So that means we have a factor of 4 increase in density through the shock. But the other quantities can change more:

$$P_2 \approx \frac{2\gamma}{\gamma+1}\mathcal{M}^2 P_1 = \frac{3}{4}\rho_1 v_1^2$$

$$T_2 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} T_1 \mathcal{M}^2 = \frac{3}{16}\frac{\bar{m}}{k_B} v_1^2$$

So in the rest-frame of the shock, the post-shock kinetic energy is:

$$\frac{1}{2}v_2^2 \approx \frac{1}{32}v_1^2$$

and the post-shock thermal energy is:

$$\frac{3}{2}\frac{k_B T_2}{\bar{m}_2} \approx \frac{9}{32}v_1^2$$

So roughly half of the pre-shock kinetic energy is converted to thermal energy.

A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas.

XXII.1.7 Blast Wave (again)

Blast wave solution:

1. free expansion (few $\times 10^{3-4}\,\mathrm{km\,s^{-1}}$, supersonic)
2. Sedov (sweeps up mass comparable to ejecta mass): energy driven
3. radiative/snow-plow: momentum driven (radiate away energy, but always conserve momentum)

XXII.1.7.1 Free Expansion

mass of ejecta $\gg$ swept up mass. $M_{\rm ej} \sim 1\,M_\odot$, and:

$$\frac{1}{2}M_{\rm ej}v^2 = 10^{44}\,\mathrm{J}$$

gives $v \approx 10^4\,\mathrm{km/s}$, which means temperatures in the X-ray regime. This expands until it sweeps up a comparable mass in the ISM with $n \sim 10^6\,\mathrm{m^{-3}}$ or $\rho = 1\times10^{-21}\,\mathrm{kg/m^3}$. Swept up mass is $(4/3)\rho_{\rm ISM}r^3$:

$$r = \left(\frac{3M_{\rm ej}}{4\pi\rho}\right)^{1/3} \approx 2\,\mathrm{pc}$$

XXII.1.7.2 Sedov

Cooling is slow. The shock has swept up a lot of mass, but it can't get rid of energy through radiation yet.

We have ρ_1 outside the shock, and ρ_2 inside. We know that $\rho_2/\rho_1 \approx 4$ since it is a strong shock.

Look at the total energy of the expanding shock wave. It has a radius R_s. The energy is:

$$E \sim \left(\frac{4}{3}\pi R_s^3\rho_1\right)v_s^2$$

(the mass of the swept up stuff times the shock velocity squared). Assume $v_s \sim R_s/t$. So then we have:

$$E \sim \rho_1 R_s^3 \times R_s^2 t^{-2}$$

ignoring the constants. We assume that the solution always looks the same, but it just scales in overall size. We call this *self-similar*. We can then get:

$$R_s = \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}$$

and

$$v_s \propto t^{-3/5}$$

In 1950, G.I. Taylor used dimensional analysis to estimate the relationship between the energy input of an extremely powerful explosion and the growth of the resulting fireball.

Taylor used a declassified photograph of the fireball of the first atomic bomb test to calculate the yield of the bomb. He arrived at an accurate value of 20 kilotons (a total mass-energy conversion of about one gram). Taylor's publication caused some consternation at the time since the yield was still classified.

We can do the same thing with supernova explosions, looking at the shock wave to estimate the age and energy yield.

XXII.1.7.3 Radiative Phase

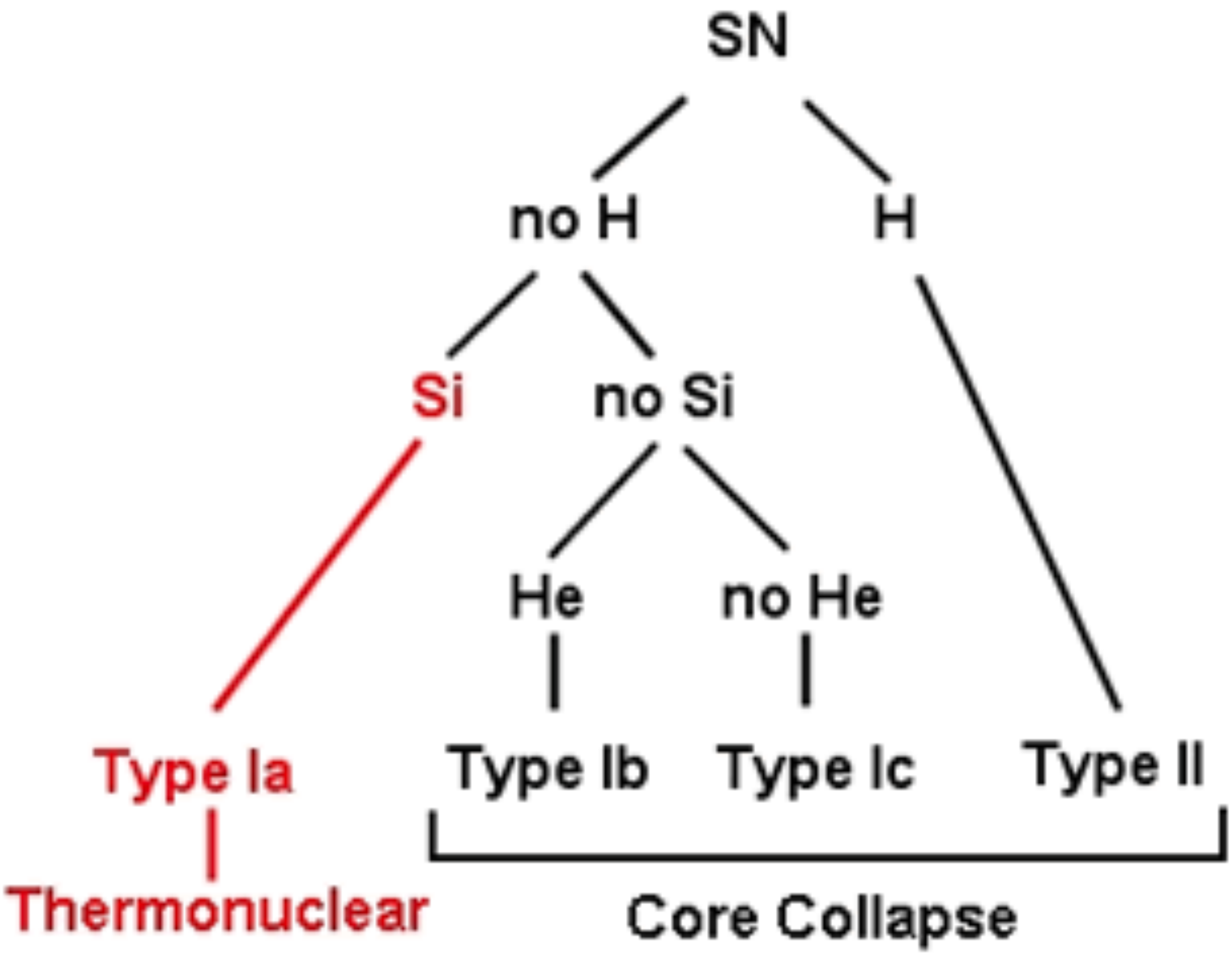
Mv =constant:

$$\frac{4}{3}\pi R_s(t)^3 \rho_1 \frac{dR_s(t)}{dt} = \text{constant}$$

The shock is cooling here ($< 10^6$ K) since more of the ions can capture electrons and cool efficiently, and we see more of the light emitted in the optical regime.

Lecture XXIII Supernova Classification

Based on appearance of spectra:



First decide based on hydrogen (I vs. II). Then check for Si. If Si, then it's a completely different type of event (thermonuclear, which we will talk about later). If no Si, then check for He.

There are subsets of Type II based on specific appearance: IIP (plateau), IIn (narrow), IIL (linear). Some of these are still debated.

For I, we see no H. What does this mean? Ia are seen in all types of galaxies: old and young. But Ib/Ic are seen mostly in young galaxies with lots of star formation. They point to massive/young stars. We are pretty sure that they still are core-collapse events, and that they lack hydrogen because the progenitors were **Wolf-Rayet** stars: massive stars that expelled their hydrogen envelopes, just leaving the helium (or lower) layers exposed.

XXIII.2 Type Ia

Thought to involve explosion of white dwarf. **Not** core-collapse of massive star. Some evidence for this is that they are seen in young and old galaxies: not tied to massive star formation.

Important for cosmology (next semester) since standard(izable) candle.

Somehow the WD explodes. Not sure how:

- Single degenerate: 1 WD + 1 other star. matter accretes onto the WD, pushes it up near the Chandrasekhar limit. Nuclear fusion ignites. Probably present in some Ia's, but $< 20\%$ (we don't see enough progenitor systems, i.e., WD+other star binaries)
- Double degenerate: 2 WDs. They merge, or the less-massive (bigger) WD accretes onto the more massive (smaller). Much harder to see before merger.

XXIII.2.1 Light Curve

Ia SN is from runaway thermonuclear fusion. The spectrum we see is not from what most of the fusion products are: they are buried deep within the expanding shell where it is optically thick. Only later can we see iron-peak elements that were produced.

Fusion makes many massive elements, but they are ultimately not stable. Optical emission is powered first by:

$$^{56}_{28}\mathrm{Ni} + e^- \rightarrow ^{56}_{27}\mathrm{Co} + \nu_e + \gamma$$

Electron capture with half-life of 6 days. This energy goes into the optically-thick shell. Then this itself decays:

$$^{56}_{27}\mathrm{Co} + e^- \rightarrow ^{56}_{26}\mathrm{Fe} + \nu_e + \gamma$$

with half-life of 77 days (also spontaneous decay).

1. Ejecta for opaque, expanding sphere
2. Energy comes from radioactive decay of Ni and Co at declining rate
3. Early:
   - High opacity: energy goes into KE
   - Small L from small, optically-thick shell
4. Pre-maximum:
   - Ejecta become more dilute
   - Diffusion time comes close to time since explosion
   - L increases

5. Maximum:
    - Energy emitted L keeps increasing while energy produced keeps declining
    - "Old" photons leak out
6. Decline:
    - L is equal to rate of energy production
    - Mostly powered by Co (Ni has decayed)

Measuring the total light emission tells us how much Fe was produced, which we can also look at from the luminosity at the peak:

$$L_{\rm Ni} \approx \dot{S}(t_{\rm max}) M_{\rm Ni}$$

$\dot{S}$ is the rate of energy generation by radioactive decay:

$$\dot{S} = 7.74 \times 10^{36} e^{-t/8.8\,{\rm day}} + 1.43 \times 10^{36} e^{-t/111\,{\rm day}}\,{\rm W}\, M_\odot^{-1}$$

This then allows us to estimate how many heavy elements were produced through the universe by Ia supernovae. The mix of elements for Ia are different than from Type II:

- Ia: Fe peak elements (Fe, Cr, Mn, Co, Ni)
- II (core-collapse): α elements (C + $x \times \alpha$ to get Ne, Mg, Si, S, Ar, Ca, Ti)

Lecture XXIV Stellar Pulsations

In 1595 amateur astronomer say huge changes in brightness from the star o Ceti. Found period of 11 months, named “Mira” (“wonderful”).

In 1784 δ Cephei was found to be variable with smaller changes and period 5 d. Many similar stars called classical Cepheids are now known.

Many of these were found and cataloged by Henrietta Leavitt at Harvard. Moreover, she discovered that from all of the Cepheids in the SMC there was a relation between period and luminosity (SMC means fixed distance): brighter means longer period. This could mean a way to figure out distances to many distant stars and galaxies (very important to Hubble). This has now been extensively refined by including near-infrared measurements and additional information to find a tight period-luminosity relation for Cepheids.

At first many explanations were put forward to explain these stars. Binaries, star-spots, etc. Many of these were eventually discarded in favor of radial pulsations: the star is breathing. You can see this by looking at brightness along with temperature and velocity: most of the change in brightness ($\sim R^2T^4$) is from a change in T, although the size does change as well.

When looking at many types of stars, see similar pulsation behavior across many different luminosities but all at similar T. This is the instabillity strip: mostly vertical, showing same physics across a wide range of stars. The different sources in this region each have different names depending on the first one of each type to be discovered. There are also other types of variable stars which have a different mechanism.

XXIV.2 Physics of Pulsations

Just like with Earth, seismology can tell us a lot about the interior of a star.

First, try to estimate period Π from properties. How long will it take a sound wave to cross the star? Assume

$$c_s = \sqrt{\frac{\gamma P}{\rho}}$$

is the adiabatic sound speed. We can use HSE as a background condition to find:

$$P(r) = \frac{2}{3}\pi G\rho^2(R^2 - r^2)$$

which is for a $n = 0$ polytrope with constant density ρ (obviously just an approximation). So we can integrate:

$$\Pi \approx 2\int_0^R dr\,\frac{1}{c_s} = \int_0^R \frac{dr}{\sqrt{\frac{2}{3}\pi\gamma G\rho(R^2 - r^2)}}$$

This isn’t fun, but can be done to find:

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

As before, $\Pi \propto 1/\sqrt{G\rho}$, with only the constants out in front to sort out.

If you use this, you find for smaller stars (higher densities) that they should have smaller periods. This helps explain the period-luminosity relation: since the instability strip is at constant T and the variation in mass is small, most of the variation along the strip is just in R, so that explains a variation in ρ and hence Π.

For the pulsations we are talking about here, they are standing waves with radial motion of the gas ("radial modes"). This can move the whole star at once (fundamental) or have one or more radial nodes where the gas is not moving (harmonics or overtones). Most stars pulse in the fundemental, but some do one of the harmonics or even more than one mode.

Think about a variable star as a series of heat engines, one for each layer. They do work when expanding or contracting depending on PdV. When averaged over a cycle this can be positive (net work) or negative (absorb energy). If the sum of all layers is positive, the pulsations will grow in amplitude. But where can this driving happen?

Eddington suggested a "valve" to drive pulsations. Imagine the star has contracted a bit, so it is smaller and hotter. If the opacity were to increase as this happened, then it would dam up the energy flowing out, which would push the layers past where they started. Then the layers would be more transparent and the trapped energy could escape. This is called the κ mechanism (for opacity).

This is not normally the case. For Kramer's law, $\kappa \propto \rho/T^{3/5}$. Since T increases on compression as well, overall κ goes down when compressed, which would damp pulsations. But there are special regions/circumstances where this can work.

It happens where H or He is partially ionized. If the gas is neutral or fully ionized, adding energy (heat) will cause temperature to change, which damps pulsations. But if it is near the transition from one ionization state to another the energy will just cause a change in the number of ions at a fixed T (a phase change). So ρ can change without T changing, and the opacity can change in the right direction.

This happens in H and He partial ionization zones. First is H I→H II or He I→He II or , where $T \sim 10^4$ K at the location of the zone. Other is He II→He III at $T \sim 4 \times 10^4$ K. Just where these zones are within a star determine if we see pulsations. If star is too hot > 7500 K or so, then they will be very close to surface with low density and too little mass to make big pulsations. This determines the "blue edge" of the zone. If the star is too cool you can get convection that damps pulsations. Most of the actual driving is done by the He II zone.

XXIV.3 Linearized Hydrodynamics

We modeled stars in equilibrium before, and talked about hydrodynamics to model shocks. For pulsations we can look at linearized equations that are close to but not exactly at equilibrium. Let us look at a one-zone model: mass of star M is all at central point, with a thin spherical layer at radius R with mass m. The rest of the star is just massless gas with pressure P. We can take the hydrodynamics equation for momentum we derived before and re-write as:

$$m\frac{d^2R}{dt^2} = -\frac{GMm}{R^2} + 4\pi R^2 P$$

which is just a form of Newton's second law. In our linearized model, variables like $P(t)$ will be $P_0 + \delta P(t)$, with P_0 constant and $\delta P(t)$ small. Then in our work we can ignore things with multiple powers of $\delta P(t)$ since they will be even smaller.

So the equilibrium part of this will be:

$$\frac{GmM}{R_0^2} = 4\pi R_0^2 P_0$$

We can write $P = P_0 + \delta P$, $R = R_0 + \delta R$ and we find:

$$m\frac{d^2(R+\delta R)}{dt^2} = -\frac{GMm}{(R_0+\delta R)^2} + 4\pi(R_0+\delta R)^2(P_0+\delta P)$$

We can expand this a bit:

$$\frac{1}{(R_0+\delta R)^2} \approx \frac{1}{R_0^2}\left(1 - 2\frac{\delta R}{R_0}\right)$$

and keeping only terms linear in a δ we find:

$$m\frac{d^2\delta R}{dt^2} = -\frac{GmM}{R_0^2} + \frac{2GMm}{R_0^3}\delta R + 4\pi R_0^2 P_0 + 8\pi R_0 P_p \delta R + 4\pi R_0^2 \delta P$$

We can cancel out the equilibrium piece in the first and third terms on the RHS. So we get:

$$m\frac{d^2\delta R}{dt^2} = \frac{2GMm}{R_0^3}\delta R + 8\pi R_0 P_0 \delta R + 4\pi R_0^2 \delta P$$

Which is good, but we still have too many variables (δP and δR). We can reduce these through an equation of state. In this case we will assume adiabatic: $PV^\gamma = \mathrm{constant}$. This means no energy entering or leaving: the pulsations are locked in place.

Since the volume of our model is $\frac{4}{3}\pi R^3$, we have $PR^{3\gamma} = \mathrm{constant}$. We can take derivatives to get:

$$\frac{\delta P}{P_0} = -3\gamma\frac{\delta R}{R_0}$$

so we can eliminate δP. We can also say that $8\pi R_0 P_0 = 2GMm/R_0^3$, which allows us to cancel the m terms. So we get:

$$\frac{d^2(\delta R)}{dt^2} = -(3\gamma - 4)\frac{GM}{R_0^3}\delta R$$

We can recognize this as a differential equation for a harmonic oscillator. If $\gamma > 4/3$ then the RHS will be negative, and the oscillator will oscillate with frequency:

$$\omega^2 = (3\gamma - 4)\frac{GM}{R_0^3}$$

From this we find

$$\Pi = \frac{2\pi}{\sqrt{\frac{4}{3}\pi G\rho_0(3\gamma - 4)}}$$

which is very similar to before.

We can also see that if $\gamma \lesssim 4/3$, then the RHS will be > 0 and the solution will be an exponential. So the star will grow out of bounds or collapse. This is similar to the stability criterion we derived before based on the Virial theorem.

To do this properly (of course), we want to run it on a computer.

XXIV.4 Will It Propagate?

Another question we can ask is what ranges of frequencies will propagate in what types of stars. There is an analogy to seismic waves in the Earth.

Consider a layer of the atmosphere between r and $r+dr$ with density $\rho(r)$. At r the acceleration of gravity is $g = GM/r^2$, so the weight of the layer is $g\rho(r)dr$. If the weight is equal to the pressure gradient $P(r) - P(r + dr)$ then the layer is in HSE. In this case if it is then $\rho(r) = \rho(0)e^{-r/H}$, where H is the scale-height, $H = k_B T/\bar{m}g$.

If a pressure wave is propagating radially, it will displace the layer. This will lead to small changes in ρ and P, denoted by $\rho'(r,t) - \rho(r)$ and $P'(r,t) - P(r)$. This is similar to the $\delta\rho$ and δP that we considered before. Let $\xi(r,t) = \delta R$ be the displacement at t for the particles at r. So for the particles at $r + dr$ the displacement will be:

$$\xi(r + dr, t) = \xi(r,t) + \frac{\partial \xi}{\partial r} dr$$

So if a particle started at t in a layer of thickness dr, it is not in a layer of thickness $(1 + \partial\xi/\partial r)dx$, so this thickness can change fractionally by $\partial\xi/\partial r$.

If the thickness changes, then the density decreases. For infinitesmal changes we can relate these by:

$$\frac{\delta\rho}{\rho_0} = -\frac{\partial\xi}{\partial r}$$

If no heat energy is added or taken away from the layer (adiabatic), then we can also say:

$$\frac{\delta P}{P_0} = \gamma \frac{\delta\rho}{\rho_0}$$

So we now need to check the forces, going back to Newton's second law:

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial}{\partial r}\delta P$$

which is very similar to the version we had before, but now written in terms of density ρ instead of mass m.

We combine these equations to write:

$$\frac{\partial^2 \xi}{\partial t^2} = c_s^2 \frac{\partial^2 \xi}{\partial r^2} - \frac{c_s^2}{H}\frac{\partial \xi}{\partial r}$$

with $c_s = \sqrt{\gamma P/\rho}$ as usual.

So if $H \to \infty$ (uniform density), then we just have a wave equation for a propagating sound wave. Any wave can propagate.

But with finite H, we can still look for oscillatory solutions with angular frequency ω. We assume that the energy density of the wave $\rho\omega^2\xi/2$, will be constant, so we guess:

$$\xi(r,t) = \frac{\Xi(r)}{\sqrt{\rho(r)}} e^{i\omega t}$$

(the assumption informs the denominator). From this we get a differential equation for $\Xi(r)$:

$$\frac{d^2\Xi}{dr^2} + \frac{\omega^2 - \omega_c^2}{c_s^2}\Xi = 0$$

with $\omega_c = c_s/2H$ is the cutoff frequency.

So we can get a solution $\Xi(r) = Ae^{-ikr}$ with wavenumber:

$$k = \pm\sqrt{\frac{\omega^2 - \omega_c^2}{c_s^2}}$$

So we can find $k > 0$ which means waves moving upward, we have two solution types:

- $\omega > \omega_c$:
$$\mathrm{Re}\xi(r,t) = \frac{A}{\sqrt{\rho(r)}} \cos(\omega t - kr)$$
- $\omega < \omega_c$:
$$\mathrm{Re}\xi(r,t) = \frac{A}{\sqrt{\rho(r)}} e^{-\chi r} \cos \omega t$$

with $k = \sqrt{(\omega^2 - \omega_c^2)/c_s^2}$ and $\chi = \sqrt{(\omega_c^2 - \omega^2)/c_s^2}$.

So in the first case the wave propagates upward with growing amplitude (since $\rho(r)$ is decreasing) but constant energy density. But in the second case the oscillation is killed by the $e^{-\chi r}$ exponential: the wave is evanescent.

So we have a cutoff frequency $\omega_c = c_s/2H$, which is the minimum frequency for a pressure wave to propagate in an isothermal atmosphere. Since $c_s = \sqrt{\gamma P/\rho} = \sqrt{\gamma k_B T/\bar{m}}$ and $H = k_B T/\bar{m}g$, can expect that $\omega_c \propto 1/g\sqrt{T}$. Which means that higher up (g and T decrease) ω_c increases and fewer waves can propagate.

We could do a more complicated version and find:

$$\omega_c^2 = \frac{c_s^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right)$$

if the scaleheight (which can be defined even in a non-isothermal atmosphere) changes with radius. Similar analyses can be done for other kinds of waves. For instance, we considered pressure to be the restoring force here. But could also have gravity be the restoring force, and then we have gravity waves which behave differently (opposite of convection).

Overall this leads to behaviors where waves can reflect, dissipate, refract, etc. We have to consider modes that are non-radial as well.

Lecture XXV Telescopes

Why do we use telescopes? How big a telescope do you need?

photons come at some rate: source has a flux density F_λ in J/s/m^2/Å. To get rate of photons:

$$F_\lambda \times \frac{\text{telescope area}}{\text{photon energy}} \times \text{filter width} \times \text{efficiency}$$

gives N (photons/s). Number detected is $N\times$time:

$$n = F_\lambda \frac{A}{hc/\lambda} \Delta\lambda \Delta t \eta$$

e.g., magnitude 20 (faint for eyes, not for telescopes) star: $F_\lambda = 3.6 \times 10^{-20} \mathrm{W\,m^{-2}\,\text{Å}^{-1}}$ (by eye can see mag$=6$, so this is $10^{-(20-6)/2.5} = 2.5 \times 10^{-6}$ times as bright). Use:

- $\lambda = 500\,\mathrm{nm}$, so $hc/\lambda = 4 \times 10^{-19}$ J
- diameter $D = 10\,\mathrm{m}$, so area $A = 78\,\mathrm{m}^2$
- efficiency $\eta = 20\%$
- width $\Delta\lambda = 1000\,\text{Å}$

gives 1.4×10^3 photons/s. How long do we need to observe for? How many seconds are enough?

XXV.2 Poisson Statistics

for counting

expect r events/s, wait t seconds. So we *expect* $rt = n$ events (cars, raindrops, people, etc.). How many actually come? Probability that we see m when expect n:

$$\frac{e^{-n} n^m}{m!}$$

- $P(0) = e^{-n}$
- $P(1) = ne^{-n}$
- $P(2) = n^2 e^{-n}/2$

sketch.

Expected number is $m = n$, but we often get more or less. What is important here is width: we don't always see exactly as many as we expect, but we want to know how close we will come on average. In general, 68% of the time we see somewhere in $n \pm \sqrt{n}$ (this is 1-σ, central limit theorem, $n \gg 1$). 95% of the time we see $n \pm 2\sqrt{n}$. 99.7% of the time we see $n \pm 3\sqrt{n}$.

So if we want to be *sure*...

We want to see enough photons that we can be sure that there is something real there. Usually we say $> 3\sigma$ confidence, so only wrong 1 time in 1000. $n/\sqrt{n} = 3$, so $n > 9$. That means we need 9 photons per exposure, so we can have $\Delta t = 6\,\mathrm{ms}$ (as I said, this is very easy for a telescope).

But objects can be a lot fainter (27th mag), can have background noise, can have lower spectral width (spectra).

XXV.3 Other Wavelengths

Optical observing: light behaves mostly like a particle. And we can do it from the ground. Does this change at other wavelengths?

XXV.3.1 Coherence

Consider number of photons in a single coherence cell:

$$\delta \approx \Delta\nu \left(\frac{F_\nu}{h\nu}\right) \Delta\tau A_c$$

with $\Delta\tau$ the coherence time, $\Delta\nu$ the coherence bandwidth, and A_c the coherence area. Based on the uncertainty principle $\Delta\tau\Delta\nu > 1$ and $A_c \approx \lambda^2 = (c/\nu)^2$, so we get:

$$\delta \approx \frac{F_\nu}{h\nu}\left(\frac{c}{\nu}\right)^2 = c^2\frac{F_\nu}{h\nu^3}$$

From a blackbody,

$$F_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

So

$$\delta \sim \frac{2}{e^{h\nu/kT} - 1}$$

For the Sun at $T = 6000\,\mathrm{K}$, at optical wavelengths $\lambda = 500\,\mathrm{nm}$ we find $\delta = 0.02$. This is $\ll 1$, so it behaves like a particle.

At radio wavelenghts $\lambda = 1\,\mathrm{m}$, $\delta = 8 \times 10^5 \gg 1$ so it behaves like a wave. $\delta = 1$ at a wavelength of $2\,\mu\mathrm{m}$.

XXV.4 Radio Telescopes

$\lambda > 1\,\mathrm{mm}$ or so, can know phase of wave. Atmosphere is transparent up to wavelengths of $10\,\mathrm{m}$ (beyond that is blocked by the ionosphere).

Do we need dark skies? No: need free from interference:

- 90–100 MHz: $\lambda = 3\,\mathrm{m}$, FM band
- 1 GHz (30 cm): WiFi, phones, microwaves, etc.

RFI. Better in valley than on mountain.

Telescopes can be anything from "light buckets" to coat hangars. Need a lot of area.

The surface needs to be smooth to $< \lambda/4$. Optical: 100 nm (polished glass, heavy & expensive). Radio: 1 cm (chicken wire).

So we can make things much bigger, which is good since sources are faint.

D up to 300 m (Arecibo). To detect, no longer look at individual photons, but treat like a wave.

XXV.5 Seeing

Optical: resolution $> 1''$ by *seeing*: turbulence in the atmosphere. To do better: go to top of mountain, go to space (expensive), correct for turbulence (hard). Last is *adaptive optics*. Use laser guide stars: make a (fake) perfect star, then see how it gets distorted. Can compute how to undo that.

XXV.6 Resolution

Diffraction limit. We can only see things that are bigger than λ/D. In the optical this is limited in any case to $> 1''$. compare to Arecibo: $49''$ at 6 cm, so much worse. And we can't make a dish (much) bigger. But luckily we don't need to:

use multiple dishes (interferometer). Then resolution is λ/B where $B \gg D$. Need to combine as a wave (maintain phase). Area (signal-to-noise) limited by D (or area$= N \times D^2$), but resolution from B.

- Very Large Array: 27 dishes, $D = 25\,\mathrm{m}$ each. B up to 30 km, so total A is $1/9$ Arecibo by θ down to $1''$ or better
- Very Long Baseline Array: 10 dishes, 25 m, B up to continent (8000 km using islands). $\theta = 1\,\mathrm{mas}$

Do interferometers work at other wavelengths? Yes, but it's hard.

- IR: Keck, VLT, some others. Up to 4 telescopes, for specialized projects, within a few 10's of m. Need to know spacing between telescopes to $\lambda/4$ which is very hard.
- Others: eventually

- lion at 10 km: $100''$
- movie screen at the Moon: $0.01''$

XXV.7 Infrared

targets: warm things. dusty things (blocks optical light, re-emits as IR). Proto-stars, star-forming galaxies, gas clouds.

$\lambda < 5\,\mu$m from ground. After that atmosphere absorbs, need to go to space. Better from high mountain in any case.

Issues: Wien displacement, peak of BB in the IR from anything warm (300 K is 10 μm). So telescopes, people, sky all make "noise." Solution is to make the whole telescope cold, ideally in space.

XXV.8 UV/X-ray

Targets: hot thigs. WD, BH, NS.

atmosphere blocks so we mostly need to go to space. X-rays are even harder: cannot make a mirror to focus the light. Have to use *grazing incidence* or other tricks.

XXV.9 Astronomy Without Photons

neutrinos, cosmic rays, gravitational waves. All hard, in infancy.

ν and CR: indirect detection. e.g., ICECUBE. 1 km^3 of ice at south pole. strings of light detectors. When ν passes through, mostly goes on without interacting. Occasionally hits proton, generates e or μ. When it does they will be traveling faster than local c (not c in vacuum). This makes Cerenkov light, or a blue flash (sort of like a shock wave).

Lecture XXVI Extra Solar Planets

Real revolution since 1995.

XXVI.2 How To Find Them?

XXVI.2.1 Can we see them directly?

Reflected light: fraction of the star's light intercepted by planet is

$$\frac{\pi R_p^2}{4\pi a^2} \approx 10^{-8}\left(\frac{a}{1\,\mathrm{AU}}\right)^{-2}$$

This is not very much. It isn't that planets are so faint, but that the stars are so much brighter and so close.

Planets are also warm, so they emit their own thermal radiation:

$$\frac{\pi R_p^2 T_p^4}{\pi R_s^2 T_s^4} \sim 10^{-6}$$

All this is done at an angular separation:

$$0.1\,\mathrm{arcsec}\left(\frac{a}{1\,\mathrm{AU}}\right)$$

at a distance of $10\,\mathrm{pc}$. But this has been done, and better instruments are making it easier.

XXVI.2.2 Radial Velocities

The main method, and the most secure, although *Kepler* is changing that.

$$\Delta v = 30\,\mathrm{m/s}\left(\frac{a}{1\,\mathrm{AU}}\right)^{-1/2}\left(\frac{M_p}{M_J}\right)$$

Can get down to precisions of 30 cm/s, so can find small planets. Different planets are sinusoids of different amplitudes/periods/phases, so can find multiple planets (Fourier analysis).

However, there is a question of unknown inclination.

XXVI.2.3 Astrometry

See the wobble of a star back and forth, across the sky not along the line-of-sight (RV).

$$\Delta\theta \sim 0.1\,\mathrm{mas}\left(\frac{a}{1\,\mathrm{AU}}\right)\left(\frac{M_p}{M_J}\right)\left(\frac{d}{10\,\mathrm{pc}}\right)^{-1}$$

unlike the others, this depends on d a lot. This has not really been useful yet.

XXVI.2.4 Transits

This is the *Kepler* revolution. See light blocked by the planet.

Depth is:

$$\frac{\pi R_p^2}{\pi R_s^2} \sim 0.01$$

(only if edge-on). The depth, length of an eclipse tell you about the radius of the star. If you can also measure the RV curve you can get the mass, and figure out mass and radius and hence composition.

Chance of transit:

$$\sim \frac{R_s}{a} \sim 0.05 \left(\frac{0.1\,\mathrm{AU}}{a}\right)$$

XXVI.3 Properties

Various databases, apps. exoplanets.eu

Large range in mass. Easier to detect high M, up to $10\,M_J$. Getting lower all the time, $N \propto 1/M$.

In periods, at the short end orbits are very close, a few days (easier to detect): makes hot Jupiters.

long end is limited by length of experiment, pushing to years.

see circular ($e = 0$), especially at low a (circularized by tides). But at higher a can be eccentric.

XXVI.3.1 Hot Jupiters

Temperature of the planet if heated by star:

Energy absorbed per second:

$$L_{\mathrm{abs}} = \frac{\pi R_p^2}{4\pi a^2}(1-A)L_s = \frac{\pi R_p^2}{4\pi a^2}(1-A)4\pi R_s^2 \sigma T_s^4$$

A is albedo, reflectivity. If $A = 1$, then a mirror.

In equilibrium, balance with emission:

$$L_{\mathrm{em}} = 4\pi R_p^2 \sigma T_p^4$$

or

$$T_p = \left(\frac{R_s}{2a}\right)^{1/2} T_s (1-A)^{1/4}$$

Can use this to get rough temperatures of the planets in the Solar System. Find that some (Earth, Venus) are hotter than expected: greenhouse effect, traps thermal radiation.

For hot Jupiters, $T_p \approx 1000\,\mathrm{K}(0.1\,\mathrm{AU}/a)^{1/2}$, compare to 120 K for Jupiter. And we see these down to $a = 0.01$ AU (Mercury is at 0.4 AU).

XXVI.3.2 Composition

Get mass from RV, R from transit. Many are larger, less dense than Jupiter. See a factor of 4 in R at the same M. Are they bloated by being hot? Doesn't quite seem so. Maybe they just never had a chance to shrink down?

XXVI.3.3 Detecting Atmospheres

Look at spectra taken during transit. The atmosphere will block certain spectral lines preferentially, compared to just rock. So during transit see depth of e.g., Na absorption is higher.

XXVI.3.4 Migration

Why do we see hot Jupiters, when the gas giants in our system are so far out?

We think they have to form out past the *frost line*: where the equilibrium temperature is cold enough for ices/volatiles to condense ($\approx 150\,\mathrm{K}$). Otherwise you will only get rocky planets.

Somehow they likely migrate inward. What process controls this? We do not know. There are a few possibilities, such as interaction between multiple planets or a planet and a disk. But this is hard to control: it cannot be too fast (the planet hits the star) or too slow.

Lecture XXVII Gravitational Light Bending

This is one of the major predictions of GR. But we can do an approximate Newtonian version first.

Consider deflection of a massive particle m around a larger mass M. The "impact parameter" is b, and the velocity at infinity is v. The particle is deflected by an angle α.

We work in the impulse approximation: $\alpha \ll 1$, so the trajectory is almost entirely straight.

Look at how much momentum is transfered to the test mass:

$$\Delta p_\perp = GMm \int_{-\infty}^{\infty} dt \frac{b}{(b^2 + v^2 t^2)^{3/2}}$$

While it looks messy, we can do this integral, and we find:

$$\Delta p_\perp = \frac{2GMm}{vb}$$

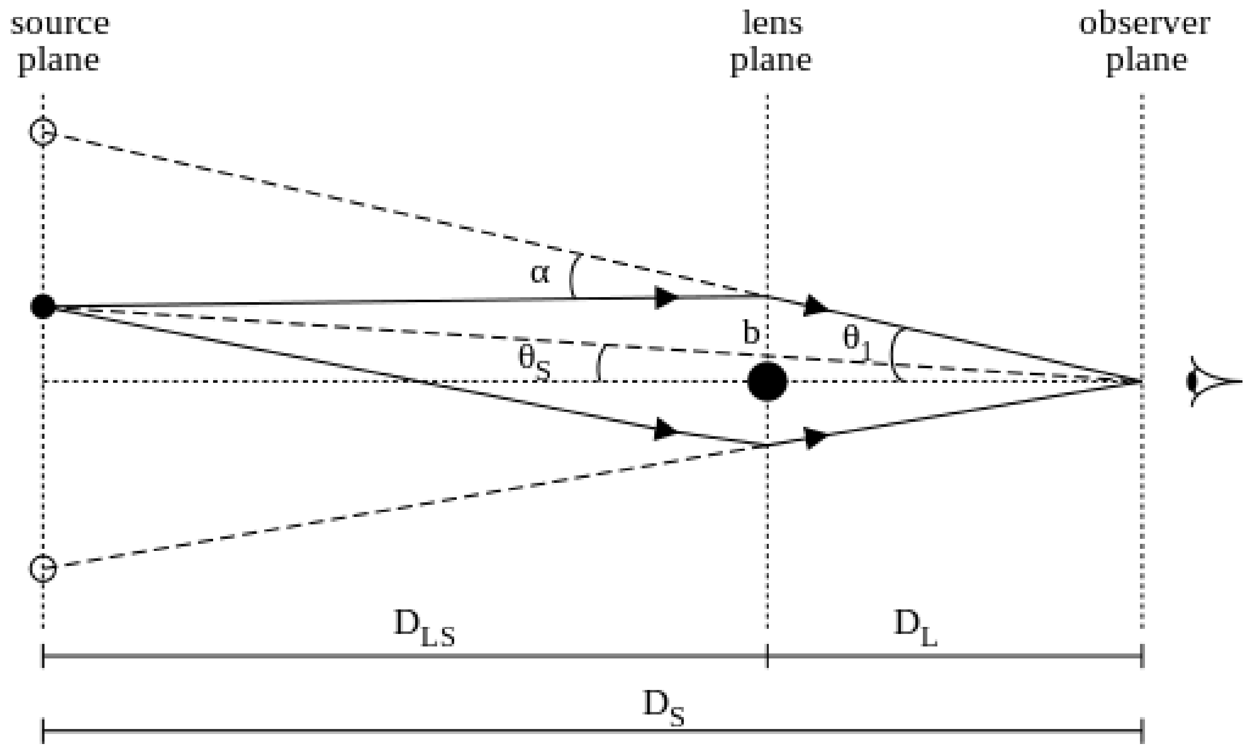
So this is the change in the momentum of the particle perpendicular to its initial motion. The angle of deflection is then just how this momentum compares to the momentum along the initial motion, mv:

$$\alpha = \frac{\Delta p_\perp}{mv} = \frac{2GM}{v^2 b}$$

Note that this is a useful result in its own right, and helps explain things like free-free radiation.

If we just take $v \to c$, we get $\alpha = 2GM/c^2 b$. But this is not correct in GR. The correct result is a factor of 2 higher, $\alpha = 4GM/c^2 b = 2R_{\text{Sch}}/b$. For instance, a light ray grazing the Sun is bent by an angle of 10^{-5} rad or about $2''$.

Consider observer separated from lens by distance D_L, and from source by D_S.



From the observer, the source is at the true position θ_S but it is seen at the image position θ_I. How are these related? We have $\theta_I = \theta_S + \alpha'$. So what is α'? Can use trigonometry plus small angle approximation to show that $\alpha' = \alpha(D_S - D_L)/D_S = (1 - D_L/D_S)4GM/c^2b = (1-x)4GM/c^2b$, where $x = D_L/D_S$.

Or, we can also say:

$$b = b_S + \alpha' D_L = b_S + (1-x)\frac{4GM}{c^2 b} x D_S$$

or

$$b = b_S + \frac{R_E^2}{b}$$

with

$$R_E^2 = \frac{4GM}{c^2} x(1-x) D_S$$

is the Einstein Ring radius (squared). Solving this gives:

$$b = \frac{1}{2}\left(b_S \pm \sqrt{b_S^2 + 4R_E^2}\right)$$

for a given b_S. This means there are two images for the two solutions, one on each side. In the case of $b_S = 0$ (perfect alignment) then $b = \pm R_E$ and we get a ring of images called an "Einstein Ring."

So this dealt with the shifting. What about the brightness? Intensity along a light ray is preserved, so if we see more light rays we will see a brighter source. We can write it as:

$$A = \left| \frac{b\, db}{b_S\, db_S} \right|$$

where A is the amplification, and the ratio is the image area compared to the initial source area. Solving this:

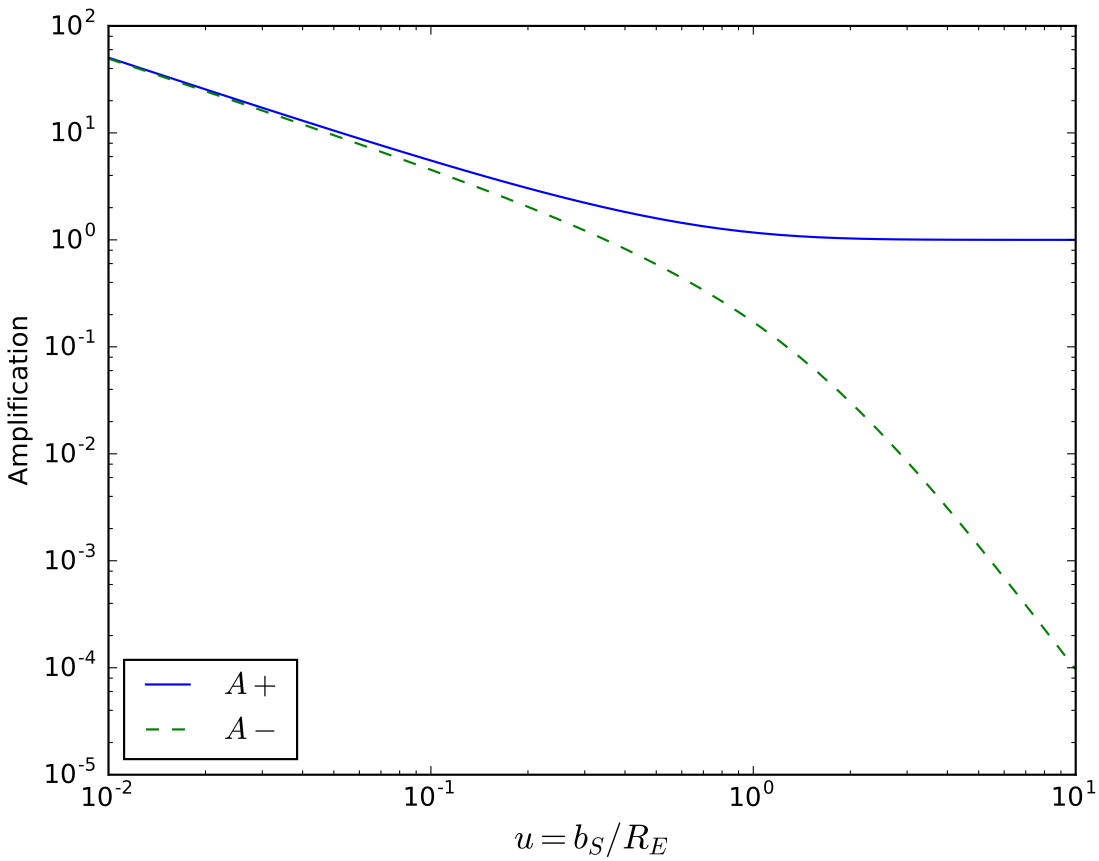
$$A_{\pm} = \left|\frac{b}{b_S}\right| \frac{1}{2}\left(1 \pm \frac{b_S}{\sqrt{b_S^2 + 4R_E^2}}\right) = \frac{1}{4}\left(1 \pm \sqrt{1 + \frac{4}{u^2}}\right)\left|1 \pm \frac{1}{\sqrt{1 + \frac{4}{u^2}}}\right|$$

with $u = b_S/R_E$. Simplifying more:

$$A_{\pm} = \left|\frac{1}{4}\left(2 \pm \frac{2 + \frac{4}{u^2}}{\sqrt{1 + \frac{4}{u^2}}}\right)\right| = \left|\frac{1}{2} \pm \frac{1}{2}\frac{2 + u^2}{u\sqrt{u^2 + 4}}\right|$$

or:

$$A_{\pm} = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \frac{1}{2}$$



We can look at three limits here:

- In the aligned case, $u \to 0$ and $A_{\pm} \to \infty$. This would suggest we have infinite magnification, but in reality the source is not infinitely small so there are parts that are mis-aligned.

- In the mis-aligned case, $u \to \infty$ and $A_+ \to 1$ while $A_- \to 0$. So we can essentially see only a single image with unity amplification, which is basically the normal non-lensed scenario.
- If $u = 1$, then $A_+ = 1.17$ and $A_- = 0.17$.

Overall, in order to see two images we want alignment to within $\sim R_E$.

How far apart are the two images?

$$\theta_I = \frac{b}{D_L} = \frac{1}{2D_L}\left(b_S \pm \sqrt{b_S^2 + 4R_E^2}\right)$$

So the difference is:

$$\Delta\theta_I = \frac{1}{D_L}\sqrt{b_S^2 + 4R_E^2} = \frac{R_E}{D_L}\sqrt{u^2+4}$$

and we mostly care about $u \lesssim 1$. This means that the separation will be

$$\Delta theta_I \approx 2\frac{R_E}{D_L} = 2\theta_E$$

where

$$\theta_E = \frac{R_E}{D_L} = \sqrt{\frac{4GM}{c^2}\frac{D_S - D_L}{D_S D_L}} = 0.9\,\mathrm{mas}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{10\,\mathrm{kpc}}{D_L}\right)^{1/2}\left(1 - \frac{D_L}{D_S}\right)^{1/2}$$

So the typical separation is $\sim 1\,\mathrm{mas}$, which is just about impossible to see with optical telescopes as we discussed earlier. However, we can see the effect of the combined magnification:

$$A = A_+ + A_- = \frac{u^2+2}{u\sqrt{u^2+4}}$$

which is always > 1.

What we are interested in here is the random lensing of a background star by some massive foreground object. In that case the lens is moving relative to the line between the source and the observer, with:

$$b_S^2 = b_{\min}^+ v^2 (t - t_0)^2$$

with v the transverse velocity. So we get:

$$u = \sqrt{u_{\min}^2 + \left(\frac{v}{R_E}\right)^2 (t-t_0)^2} = \sqrt{u_{\min}^2 + \left(\frac{t - t_0}{\Delta t_E}\right)^2}$$

with $u_{\min} = b_{\min}/R_E$ and $\Delta t_E = R_E/v$ the “Einstein Ring Crossing Time:”

$$\Delta t_E = 0.2\,\mathrm{yr}\left(\frac{200\,\mathrm{km/s}}{v}\right)\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{10\,\mathrm{kpc}}{D_L}\right)^{1/2}\left(1 - \frac{D_L}{D_S}\right)^{1/2}$$

which is the timescale over which u and A change. This ends up being the typical duration of a lensing event.

We see a characteristic lightcurve, where those with small $u_{\min}$ have larger magnifications.

How can we tell that what we see is microlensing and not random stellar variability? Lensing affects all wavelengths the same, which is rare.

What is the probability that a given star in the Galaxy is lensed with $u < 1$ at a given time? We can write this in terms of a cross section or an optical depth:

$$\tau = \int_0^L dx\, n(x)\sigma(x)$$

For the lensing problem, we can write a cross section like:

$$\pi R_E^2 = \pi \frac{4GM}{c^2} \frac{D_L(D_S - D_L)}{D_S}$$

So:

$$\tau = \int_0^{D_S} dD_L\, n(D_L) \frac{4\pi GM}{c^2} \frac{D_L(D_S - D_L)}{D_S} = \frac{4\pi G}{c^2} D_S^2 \int_0^1 dx\, \rho(x) x(1-x)$$

with $\rho = Mn$ and $x = D_L/D_S$. So we can estimate what ρ is and figure this out. If ρ is constant, then $\tau \sim 2\pi G\rho D_S^2/3c^2$. If $D_S \sim 50\,\mathrm{kpc}$ is the size of the Galactic halo which is the whole system we are considering, then:

$$\tau \sim \frac{GM_{\mathrm{total}}}{D_S c^2} \sim \frac{v^2}{c^2}$$

since by the Virial theorem, $v^2 \sim GM_{\mathrm{total}}/D_S$. So for $v \sim 200\,\mathrm{km/s}$, which is a typical speed for stars in the Milky Way, we have $\tau \sim 5 \times 10^{-7} \ll 1$. So we need to observe millions of stars to find microlensing events, and we need to look every few weeks to see changes.

There are experiments that do this. Looking for MACHOs: possible dark matter consituent that would be visible via microlensing even if it doesn't emit itself. Overall microlensing is found, but not nearly enough to account for dark matter (next semester).

Lecture XXVIII Gravitational Waves

Conservation of mass implies that GW from mass monopole is 0 (same with conservation of electric charge in EM).

Conservation of momentum implies that GW from mass dipole is also 0. This isn't the case with EM. For GW we need to go to quadrupole for the lowest order radiation. Can show:

$$h \sim \frac{G}{c^4 d}\frac{d^2 Q}{dt^2}$$

is the strain, where Q is the mass quadrupole moment. Strain is the gravitational wave amplitude. Notice that this is $h \sim 1/d$ for the amplitude. This means that the energy flux goes as $\sim 1/d^2$, which is good since that means it is conserved.

Can also work out luminosity of the waves:

$$L_{\rm GW} = -\frac{dE}{dt} = \frac{1}{5}Gc^5\left(\frac{d^3Q}{dt^3}\right)^2$$

We can say $Q \sim MR^2$ for a system with mass M and characteristic size R. The time derivatives will then be $d/dt \sim 1/T$, where T is a characteristic timescale of the system (such as orbital timescale). R/T is also a velocity, so

$$\frac{d^3Q}{dt^3} \sim \frac{MR^2}{T^3} \sim \frac{Mv^2}{T}$$

The timescale can also be related to the dynamical timescale, $T \sim 1/\sqrt{G\rho} \sim \sqrt{R^3/GM}$. So we can write:

$$L_{\rm GW} \sim \frac{G}{c^5}\left(\frac{M}{R}\right)^5 \sim \frac{G}{c^5}\left(\frac{M}{R}\right)^2 v^6 \sim \frac{c^5}{G}\left(\frac{R_{\rm Sch}}{R}\right)^2\left(\frac{v}{c}\right)^6$$

So we get a lot of energy out if the size is small compared to the Schwarzschild radius and the velocity is large compared to c. At a maximum, we get $L_{\rm GW} \sim c^5/G = 3.6 \times 10^{52}$ W, which is independent of mass. It is a huge luminosity, far greater than the EM luminosity we see.

We can also figure out the strain amplitude. Relate $Mv^2 \sim E_{\rm ns}$, which is the non-spherical part of the kinetic energy, $E_{\rm ns} = \epsilon E$. We get:

$$h \sim \frac{G}{c^4}\frac{\epsilon E}{d}$$

The factor ϵ is what fraction of the energy plays a part in emitting GWs. Combining the two, we get:

$$h \approx 10^{-22}\left(\frac{E_{\rm GW}}{10^{-4}\,M_\odot}\right)^{1/2}\left(\frac{f}{1\,\rm kHz}\right)^{-1}\left(\frac{\tau}{1\,\rm ms}\right)^{-1/2}\left(\frac{d}{15\,\rm Mpc}\right)^{-1}$$

as the strain from an event at the Virgo cluster (15 Mpc distance) which has a frequency of $f = 1$ kHz, a duration of $\tau = 1$ ms, and has $10^{-4}\,M_\odot c^2$ of energy emitted as GWs.

As a concrete example, if we have a binary system, then $Q \sim Ma^2$. We get:

$$L_{\mathrm{GW}} = \frac{8GM^2a^4\Omega^6}{5c^5} = \frac{64GM^5}{5c^5a^5}$$

where we have used Kepler's Third Law $\Omega^2 = GM/a^3$. This assumes things are circular, but if the orbits are eccentric the luminosity is higher and the orbit will decay more quickly.