Astron 300 Problem Set 10

Due: Wed Dec 8 at the beginning of class.

Homework Policy: You can consult class notes and books. Always try to solve the problems yourself; if you cannot make progress after some effort, you can discuss with your classmates or ask the instructor. However, you cannot copy other's work: what you turn in must be your own. Make sure you are clear about the process you use to solve the problems: partial credit will be awarded.
Reading: Carroll & Ostlie, Chapters 23, 6

Problem 1 Accretion of planetesimals onto a proto-Earth

The figure below illustrates the trajectory of a planetesimal (rock of negligible mass and size) as it passes near a massive object (say, a proto-earth, of mass M and radius R). We are interested in the accretion cross-section for the latter object, and how its rate of growth depends on its mass.

a. We first consider the trajectory. Let the velocity of the rock relative to the proto Earth at infinity be v_∞ and the impact parameter (the separation between its initial trajectory and a parallel line through the center of the proto-earth) be b. Gravity bends the trajectory producing a closest approach a ($a < b$). Obtain a as a function of b, M and v_∞. (Hint: assume that the motion of the rock is only affected by the gravity of the proto-Earth, and use the fact that the rocks total energy and angular momentum are conserved.)

b. The accretion cross-section $\sigma = \pi b^2$ is enhanced over the geometrical cross-section (πR^2) because of the gravitational focusing.

   (b.1) Set $a = R$ and derive σ. Write your results in terms of R, mean planet density ρ, and v_∞.

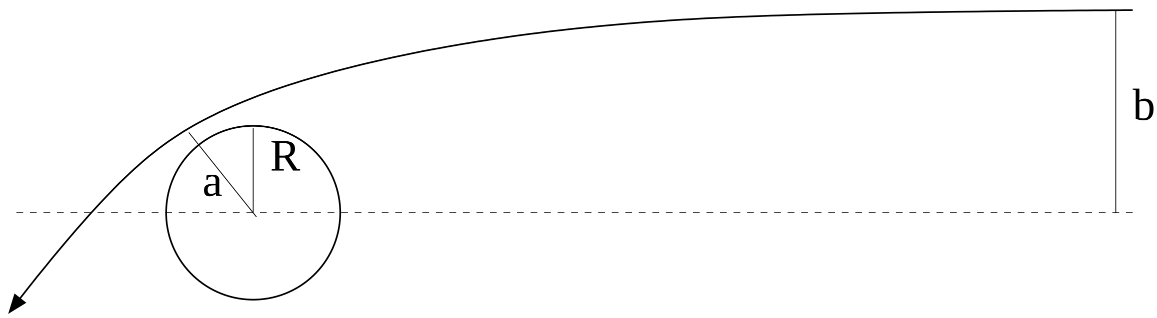
   (b.2) How large does the planet have to be for gravitational focusing to become significant? In other words, find the size $R = R_{\rm crit}$ (in terms of v_∞ and ρ) for which σ is enhanced over the geometrical value by a factor of two.

   (b.3) Calculate $R_{\rm crit}$ for $\rho = \rho_\oplus \approx 5.5 \times 10^3\,{\rm kg\,m^{-3}}$ and $v_\infty = 1\,{\rm km\,s^{-1}}$ (a small fraction of the Keplerian velocity at 1 AU).

c. The proto-earth grows in mass by accreting planetesimals, at a rate $\dot{M}$ proportional to σ.

(c.1) Write down σ in terms of ρ and M for the case that $R \gg R_{\rm crit}$.

(c.2) Show that the time needed for a proto-planet to accrete its own mass, $t_{\rm acc} = M/\dot{M}$, scales as $M^{-1/3}$. (Thus, more massive objects grow faster; the 'rich get richer' scenario in planet formation.)



Problem 2 Radioactive Dating

Radio-active rhenium (^{187}Re) decays with a half-life $\tau_{1/2} = 4.6 \times 10^{10}$ yr. The abundance for the decay product osmium (^{187}Os) rises accordingly so as to conserve the total number of nuclei of ^{187}Re and ^{187}Os. The abundances of these two elements are measured against the abundance of ^{188}Os, an isotope of Os which is not involved in any decaying process.

a. Show that the following equation is true:

$$\frac{N_{187\rm Os}(t)}{N_{188\rm Os}} = (e^{\lambda t} - 1)\frac{N_{\rm Re}(t)}{N_{188\rm Os}} + \frac{N_{187\rm Os}(0)}{N_{188\rm Os}}$$

where $N_{\rm X}(y)$ is the number of nuclei of the element X at time y (188Os for ^{188}Os, etc.), and $\lambda \equiv \ln 2/\tau_{1/2}$.

b. When a rock solidifies (from molten or vaporous forms), all elements are locked in, with initial ratios $N_{187\rm Os}(0)/N_{188\rm Os}$ and $N_{187\rm Os}(0)/N_{\rm Re}(0)$. Since different isotopes of the same element do not have different chemical behavior, different minerals in the same piece of rock likely have the same $N_{187\rm Os}(0)/N_{188\rm Os}$ value but differ in their $N_{187\rm Os}(0)/N_{\rm Re}(0)$ values due to their different chemical compositions. At the present day, one can measure $N_{187\rm Os}(t)/N_{188\rm Os}$ and $N_{\rm Re}(t)/N_{188\rm Os}$ for different minerals in the same rock. Show that $Y = N_{187\rm Os}(t)/N_{188\rm Os}$ should depend linearly on $X = N_{\rm Re}(t)/N_{188\rm Os}$, with $Y = aX + b$, and write the values of a and b in terms of the other variables.

c. The table below lists a string of measurements for X and Y (taken from Planetary Sciences, de Pater & Lissauer). Use these to determine what is the initial abundance of $N_{187\rm Os}(0)/N_{188\rm Os}$, and how long ago the rock solidified. *(Note:*

either use a least-square solver or use a graph to determine a and b. The measurements all have similar uncertainty.)

$X = N_{\mathrm{Re}}(t)/N_{188\mathrm{Os}}$	$Y = N_{187\mathrm{Os}}(t)/N_{188\mathrm{Os}}$
0.669	0.148
0.664	0.148
0.604	0.143
0.484	0.133
0.512	0.136
0.537	0.138
0.414	0.128
0.369	0.124

Problem 3 Carroll & Ostlie 6.7

Problem 4 Carroll & Ostlie 6.8

Problem 5 Carroll & Ostlie 6.12