Astronomy 299/L&S 295

Lecture I Preliminaries

Course Description: *quantitative* astronomy. Emphasis on the “why” and the “how” rather than just the “what.” There will be some math(!) and physics. There will be no calculus.

If you don’t want that, look at ASTRON 103. If you want a lab, take ASTRON 104. Here we will cover:

- Celestial mechanics
- The nature of light and its interaction with matter
- Telescopes
- The structure and evolution of single stars
- The evolution of binary stars
- The end-products of stellar evolution
- The Solar System
- Extra-solar planets
- Galaxies & quasars
- Expansion of the universe & dark matter
- The big bang

The textbook will be Kutner: *Astronomy: A Physical Perspective*

Evaluation will be:

- Weekly problem sets (50%), with the best 10 of 11 counting.
- Midterm exam (20%)
- Final exam (30%)

I.2 Set the Stage

Astrophysics mostly starts next lecture.

I.3 Physics Synopsis

The level of physics and math that you are expected to be familiar with (but not necessarily know in detail) is:

Newton's Laws : most importantly, $F = ma$

Kinetic Energy : $K = \frac{1}{2}mv^2$

Gravitation : $F = GM_1M_2/r^2$ (on the surface of the Earth, $F = gm$)

Potential Energy : $U = -GM_1M_2/r$ (from gravity; on the surface of the Earth, $U = gmh$)

Centripetal Acceleration : $a = v^2/r$

Ideal Gas Law : $PV = NRT$

Circumference of a Circle : $2\pi r$

Area of Circle : πr^2

Surface Area of a Sphere : $4\pi r^2$

Volume of a Sphere : $\frac{4}{3}\pi r^3$

Radians : $180° = \pi$ radians, $\sin(\pi/2) = 1$, $\sin\pi = 0$, etc.

Small Angles : $\sin x \approx x$ for x very small and measured in radians. Also, $\tan x \approx x$, and $\cos x \approx 1$ (**draw these**)

Scientific notation : $A \times 10^a \cdot B \times 10^b = (AB) \times 10^{a+b}$

I.3.1 Calculus

Not required. However, it is in the book. **Don't Panic!** If you see:

$$\frac{dx}{dt}$$

or

$$\int dx f(x)$$

just read around it. Or ask questions.

I.3.2 Greek

If I use a symbol you don't recognize or can't read, **ask**!

I.4 Precision

We often do not know things very precisely. So we use $\sim$ and $\approx$ and related symbols. $\sim$ is for when we know something to *an order of magnitude*. So we if we know that $x \sim 5$, we know that x is between $5/3$ and $5 * 3$, where 3 is roughly $\sqrt{10}$. This means that the possible range for x is in total a factor of 10. We will also sometimes use $\sim$ to mean *scales as*. For example, if you were to estimate the height of a person as a function of their weight (for a wide range of people), you might expect that as you double the weight, the height changes by $2^{1/3}$. We could write height$\sim$weight$^{1/3}$. There will be a lot of variation, but this is roughly correct.

$\approx$ means more precision. It doesn't necessarily have an exact definition. But generally, if we say $x \approx 5$, that means that 4 is probably OK but 2 is probably not.

Finally, we have $\propto$, which means *proportional to*. This is more precise that the *scales as* use of $\sim$. So while for a person height$\sim$weight$^{1/3}$ is OK, for a sphere (where we know that volume is $4\pi/3r^3$) we could write volume$\propto r^3$: we take this as correct, but leave off the constants ($4\pi/3$ in this case).

I.4.1 Order of Magnitude Problems

There are many problems — in Astronomy, Physics, or Life — where we know only the basics. But we want to estimate something. So we do an "order of magnitude" estimate (also known as a "Fermi problem": see `http://en.wikipedia.org/wiki/Fermi_problem`). Basically, we want to know whether something is 1, 10, or 100, but we do not care whether it is 20 or 30. These make heavy use of the $\sim$ symbol. We will come back to these.

I.4.2 Small Angles

For small angles θ, $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. We need θ to be in radians. But we also often deal with fractions of a circle. A circle has $360°$. We break each degree into 60 minute (or *arcminutes*): $1° = 60'$. And each arcminute into 60 seconds (or *arcseconds*): $1' = 60''$, so $1° = 3600''$. But we also know that 2π radians is $360°$, so we can convert between radians and arcsec. This will come up frequently: $1'' = 360 \times 3600/2\pi \approx 1/206265$ radians.

I.5 Example Problem

Consider all of the people on the UWM campus:

- How much do all of the people on the UWM campus weigh?
- How many buses would you need to transport them all?
- If they all jumped in to Lake Michigan, how much would the water level change (numbers: 300 miles long by 118 miles wide, average depth 279 feet, volume 1180 cubic miles)? What about all of the people in the world?

- 30,000 people, 60 kg each, 1,800,000 l or 1800 m^3
- 5×10^{12} m^3
- So answer is 0
- for whole world answer is 1 cm (400,000,000 m^3 total).

Lecture II Celestial Sizes, Distances, and Coordinates

II.1.1 Units

Astronomy emphasizes *natural* units ($\odot$ is for the Sun, $\oplus$ is for the Earth):

- $M_\odot = 2 \times 10^{30}$ kg (solar mass)
- $R_\odot = 7 \times 10^{8}$ m (solar radius)
- $M_\oplus = 6 \times 10^{21}$ kg (earth mass)
- $L_\odot = 4 \times 10^{26}$ W (solar luminosity or power)
- light year $= 10^{16}$ m: the *distance* light travels in one year (moving at $c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$)
- parsec = parallax second (we will understand this later) = pc = 3×10^{16} m
- Astronomical Unit = AU = 1.5×10^{11} m (distance between earth and sun)

And then we use usual metric-style prefixes to get things like kpc, Mpc, etc.

II.2 Distances

Kutner 2.6.

How far away/big are things? Use meter stick to draw centimeter, meter, 10 meters.

Chicago : 144 km

circumference of Earth : 40,000 km

distance to Moon : 380,000 km

distance to Sun : 1.5×10^{11} m = 1 AU

solar system (orbit of Neptune): 30 AU in radius

Oort cloud : 50,000 AU

Nearest star (Proxima Centauri): 1.29 pc = 4×10^{16} m

Center of Milky Way : 8.5 kpc = 2.6×10^{20} m

Andromeda Galaxy : 0.6 Mpc = 1.8×10^{22} m and so on. There is *a lot of empty space*!

How do we measure distances to stars? First step: parallax = geometry. $\tan\theta = 1\,\mathrm{AU}/d \rightarrow d = 1\,\mathrm{AU}\tan\theta$. But stars are very far away, so θ is very small. Again, if θ in radians $\tan\theta \approx \theta$ so $d \approx 1\,\mathrm{AU}\theta$ (which puts d in AU too if θ in radians).

But remember, 1 radian is $206265''$. So if θ is in arcsec now, $d = 206265\,\mathrm{AU}(\theta/\mathrm{arcsec})$. 206265 AU has a special name: it is 1 parallax second or 1 parsec (or 1 pc or 3×10^{16} m).

How big is an arcsecond? For a quarter to be $1''$ across (diameter of 25 mm) need to be 5 km away. And we can measure much smaller angles.

II.2.1 Planetary Motion and the Copernican Model

Kutner, Chapter 22.

The geocentric model for the Universe is wrong. The Earth is not the center of the solar system, the galaxy, the universe, etc. Partly this was uncovered through observations of *retrograde motion*: stars appear the same every night, but some objects (often bright ones) move relative to the stars. These are known as *planets* (literally wanderers). Mostly the planets move from West to East. Except when they don't — then they go the other way, which is called retrograde motion. This was a 2000-year old puzzle. **Mars 1994 from Astron 103 week 3**.

In the geocentric view, it was complicated and elaborate (*epicycles*). But the heliocentric view (from Copernicus) has an elegant solution. Taking inner orbits as faster (we'll see why later), we find retrograde motion happens occasionally for the planets that are further out than the Earth.

We can then define two periods:

sidereal period (or sidereal time) is the period relative to the fixed background stars. This is close to (but not exactly the same) as a year.

synodic period is the time between when planets are closest together

and we can relate these by asking how long until the planets line up again. We define the angle of the Earth $\theta_{\mathrm{E}} = 2\pi t/P_{\mathrm{E}}$ which goes around from 0 and one revolution happens at $t = P_{\mathrm{E}}$ (θ in radians). We can do the same for Venus $\theta_{\mathrm{V}} = 2\pi t/P_{\mathrm{V}}$. $P_{\mathrm{V}} < P_{\mathrm{E}}$, so θ_{V} goes faster. They line up when $\theta_{\mathrm{E}} = \theta_{\mathrm{V}} - 2\pi$: the 2π is since Venus will have gone around one extra time. So we write:

$$\frac{2\pi t}{P_{\mathrm{E}}} = \frac{2\pi t}{P_{\mathrm{V}}} - 2\pi$$

and can solve for $1/t = 1/P_{\mathrm{V}} - 1/P_{\mathrm{E}}$, and we identify t with the synodic period. For planets outside the Earth's orbit, the sign is opposite.

II.2.2 Motion of the Earth

The Earth rotates around its axis once every 24 hours. So each hour is then $360°/24 = 15°$.

This rotation is what makes the Sun and the stars appear to move over the course of a day. The star to which the Earth's axis appears to point is the North Star (Polaris): it's not a special star, we just point near it. Because of "precession", we point to different stars over the course of about 24,000 years.

This rotation can help you figure out how long you have (for example) until the Sun goes down:

- Your hand is roughly $10°$ across when you hold your arm out
- Your finger is roughly $1°$ across

And then, just like all of the planets, the Earth goes around the Sun. Each planet takes its own time. For the Earth, this is 1 year.

II.2.3 Seasons and the Changing Sky

The rotation axis of the Earth is tilted $23.5°$ to the plane of its orbit. This means that as the Earth orbits around the Sun, the Sun appears to trace a path in the sky. We call this the *ecliptic* (constellations in the ecliptic are the zodiac). The plane of the orbit extending out to infinity is the *ecliptic plane*.

During the nothern summer, the North pole of the Earth points towards the Sun. This is roughly June 21. On the northern winter solstice, what points towards the Sun? Seasons depend on what hemisphere you are in.

solstice : roughly June (northern summer) or December (northern winter), when the "sun stands still." Longest/shortest day. Flips in the southern hemisphere.

equinox : roughly March/September 21. When the night and day are of equal length.

Just like the equator on the Earth, we can extend it into the sky to make the *celestial equator*. This divides the sky into northern & southern halves. We do the same thing with the poles. The Sun crosses the celestial equator twice per year, on the equinoxes.

winter (in north) : Sun is low in the sky, days short

summer (in north) : Sun is high in the sky, days long

spring/fall : in the middle.

II.2.3.1 Weather (Hot vs. Cold)

The Sun puts out energy at a roughly constant rate: $L_\odot$. It is changing the way that we receive this energy that makes seasons. How does this work? Do we move closer to or further from the Sun?

NO!. This wouldn't work since the northern Summer = southern winter. Instead we change the area over which the Sun's energy is spread.

Draw a circle perpendicular to the path of the Sun's light with area $1\,\mathrm{m}^2$. This defines a *flux* which is power per area: $F = L/\mathrm{area}$. The total area over which the Sun's light is spread is the area of a sphere $4\pi R^2$, so at the Earth $F_\odot = L_\odot/4\pi R^2$ with $R = 1$ AU. When the Sun is high overhead (summer), this power gets spread over a patch of the ground with the same area. So each $1\,\mathrm{m}^2$ of the ground gets heated with the full $F_\odot$ of flux. This makes it hot. But in the winter the Sun is low in the sky. So the circle that is perpendicular to the Sun's light gets spread over a wide area on the ground. So we still have $F_\odot$ of flux, but it gets spread over (for example) $2\,\mathrm{m}^2$ of area. This means that each patch of the ground gets heated by less power, which makes it colder.

II.3 How bright are stars?

Kutner 2.1

Historical method: magnitudes (messy and annoying, but has its uses).

lower = brighter, *higher = fainter*. Used to be $m = 1$ is roughly the brightest. But now we have quantified this on a *logarithmic scale*:

- 2.5 mag fainter = 1/10 the brightness
- 5.0 mag fainter = 1/100 the brightness
- 10 mag fainter = 1/10,000 the brightness

We look at things from the Sun ($m = -26.8$, as it appears from the Earth) to $m \approx 30$.

But remember that we are measuring how bright things appear! This would change if they were closer/farther. We are essentially measuring *flux*: energy per time per area. Or (energy per time) per area. (energy per time) is the same as power, and we measure this in Joules per second or Watts. So we measure flux in Watts per area, or Watts per m^2.

If instead we look at total power put out (independent of where you are) that is *Luminosity*, L. L is measured in Joules/s or W. How can we go between these? Take a star of luminosity L, and surround it by a sphere with radius R. The flux is $F = L/4\pi R^2$, since the area of that sphere is $4\pi R^2$. This is the *inverse square law* for fluxes, and it should be familiar (when you go away from a lightbulb, it gets fainter, etc.).

$L_\odot = 4 \times 10^{26}\,\mathrm{W} = 1 L_\odot$. $R = 1$ AU = 1.5×10^{11} m. So the flux on the Earth $F_\odot = 1300\,\mathrm{W\,m^{-2}}$. In comparison, on Neptune at $R = 30$ AU, the flux is 1/900 as much.

So we have magnitudes. How do we tie magnitudes (as a measure of brightness) to real fluxes?

II.3.1 Absolute Magnitudes

=M=the apparent magnitude m of a star if it were 10 pc away.

From the definition of a magnitude, a change of 2.5 mag is a factor of 10 change in F. So we can take two magnitudes m_1 and m_2 and put them with two fluxes F_1 and F_2:

$$\frac{F_2}{F_1} = 10^{-(m_2-m_1)/2.5}$$

The negative sign comes from having a lower magnitude mean brighter. Or:

$$m_1 - m_2 = -2.5 \log_{10} \frac{F_1}{F_2}$$

To put in absolute magnitudes, we use the inverse square law: $F = L/4\pi d^2$. So $F_1 = L/4\pi d_1^2$ and $F_2 = L/4\pi d_2^2$, and we divide:

$$\frac{F_1}{F_2} = \left(\frac{d_1}{d_2}\right)^{-2}$$

with $d_2 = 10\,\mathrm{pc}$. So $F_1/F_{10\,\mathrm{pc}} = (d_1/10\,\mathrm{pc})^{-2}$. This is the same as:

$$\frac{F_1}{F_{10\,\mathrm{pc}}} = 10^{-(m_1-M)/2.5}$$

we can work this through to find: $m - M = 5\log_{10}(d/10\,\mathrm{pc})$ which is known as the *distance modulus*: how much being far away changes the apparent magnitude of something.

So if we know m and M, we can get d. Or if we know m and d, can get M. FOr the Sun: $m_\odot = -26.83$, $d_\odot = 1$ AU = $1/206265\,\mathrm{pc}$. We get: $M_\odot = m_\odot - 5\log_{10}(d/10\,\mathrm{pc}) = +4.74$. This is pretty modest compared to other stars (the Sun is only remarkable by being close).

Knowing what $M_\odot$ is, if we are then given the absolute magnitude of a star we can calculate its flux and actual luminosity. That is because for two objects at the same distance (10 pc in this case) the relation between magnitudes and fluxes can work with luminosities too. So we can say:

$$F = F_\odot 10^{-(M-M_\odot)/2.5}$$

$$L = L_\odot 10^{-(M-M_\odot)/2.5}$$

where $F_\odot$ is now the flux of the Sun as perceived from 10 pc away.

Lecture III Gravity & Celestial Mechanics

And finally some physics. Kutner 5.3, 5.4

Johannes Kepler: used data from Tycho Brahe to determine 3 "laws".

1. Orbits are ellipses with the Sun at a focus. Semi-major axis is a, semi-minor axis is b. The equation of an ellipse says that the distance from the planet to the Sun + the distance from the planet to the other focus is a constant.
2. Planet-sun line traverses equal areas in equal time. (**Kepler's 2nd law from Astron 103 week 3**)
3. $P^2 = a^3$, with P the period of the orbit. For the Sun, this works with P in years and a in AU.

III.2 Elliptical Motion

aphelion = far from star, *perihelion* = close to star. Eccentricity e between 0 (circle) and 1. We find $b^2 = a^2(1 - e^2)$, with perihelion at $a(1 - e)$ and aphelion at $a(1 + e)$. Planets have only slightly eccentric orbits. For the Earth, $e = 0.0167$. So maximum distance from Sun - average is $ae = 0.0167\,\mathrm{AU} \approx 400R_{\oplus}$. This means a change in the solar flux of about $\pm 3\%$.

III.3 Newton's Laws

1. Inertia
2. $\vec{F} = m\vec{a}$
3. $\vec{F}_{12} = -\vec{F}_{21}$

(here $\vec{F}$ is a force, not a flux). Also have gravitation: $F = GMm/r^2$.

For a circular orbit of something with mass m around something with mass M that is not always the sun, Kepler's third law says: $P^2 \propto r^3$, but the constant of proportionality can change: $P^2 = kr^3$. Let's derive k. $P = 2\pi r/v$, since it has to traverse a distance $2\pi r$. Putting this in gives us

$$\frac{4\pi^2 r^2}{v^2} = kr^3$$

We need a centripetal force $F = mv^2/r$ to keep the planet in orbit: $F = GmM/r^2$. We put these together and re-arrange to get $k = 4\pi^2/GM$.

III.4 Escape: Work and Energy

Potential energy $U = -GMm/r$. It depends just on the start and stop points, not the path.

Kinetic energy $K = 1/2mv^2$.

Total energy $E = K + U = 1/2mv^2 - GMm/r$. Start at r, move away until $v = 0$ at $r = \infty$. $K(\infty) = 0$, $U(\infty) = 0$, so $E(\infty) = 0$. This will always be true (no external source of work), so $K = -U$. We can write $1/2mv^2 = GMm/r$ and get:

$$v = \sqrt{2GM/r} = v_{\mathrm{escape}}$$

is the escape speed. For the Earth this is 11 km/s: if you go this fast and point up, you will escape the Earth's gravity.

III.5 Consequences of Gravity

III.5.1 Linear Momentum

If the net force on a system is 0, then momentum is conserved.

III.5.2 Angular Momentum

is conserved also.

III.5.3 Energy

is conserved also.

III.6 2 Body Problem

This is more general than what we did before, and we no longer require $m \ll M$. This applies to binary stars, binary asteroids, black holes, planets, etc. We can write.

Since momentum is conserved, the two bodies will orbit their common center of mass. The COM can move, but it will move at a constant velocity. This is defined such that:

$$m_1 r_1 = m_2 r_2$$

Both stars must go around in the same time such that the line between them always goes through the COM:

$$P = \frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2}$$

(assuming circular motion). Or $r_1/v_1 = r_2/v_2$. Combining these we get:

$$\frac{v_1}{v_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1}$$

We also have the force:

$$F = G\frac{m_1 m_2}{(r_1 + r_2)^2}$$

As before, this must be the force needed to keep an object in a circular orbit. So on object 1, $F = m_1 v_1^2 / r_1$:

$$\frac{m_1 v_1^2}{r_1} = G\frac{m_1 m_2}{(r_1 + r_2)^2}$$

Divide both sides by m_1, and use $P = 2\pi r_1 / v_1$:

$$\frac{4\pi^2 r_1}{P^2} = \frac{G m_2}{(r_1 + r_2)^2}$$

But we also define $R = r_1 + r_2 = r_1(1 + r_2/r_1)$. With the ratio of the masses intead of radii:

$$R = r_1(1 + \frac{m_1}{m_2}) = \frac{r_1}{m_1}(m_1 + m_2)$$

Put this in:

$$\frac{4\pi^2 R^3}{G} = (m_1 + m_2)P^2$$

This is Kepler's third law! It's a bit more general than for just a system like the Earth and the Sun. We often write a instead of R.

The angular momentum is constant, and it can be useful. For each object $L_1 = m_1 v_1 r_1$ etc. So overall:

$$L = m_1 v_1 r_1 + m_2 v_2 r_2$$

But we know $m_1 r_1 = m_2 r_2$, so we can write:

$$L = m_1 r_1 (v_1 + v_2) = m_1 r_1 \left(\frac{2\pi r_1}{P} + \frac{2\pi r_2}{P}\right)$$

Substitute from before: $r_1 = m_2 R/(m_1 + m_2)$:

$$L = \frac{2\pi m_1 m_2 R^2}{(m_1 + m_2)P}$$

We substitute for P from Kepler's third law to get:

$$L = m_1 m_2 \sqrt{\frac{GR}{m_1 + m_2}}$$

III.6.1 How to Use This?

A basic result that we will use over and over again is $GM = a^3(2\pi/P)^2$. For circular systems, $v = \sqrt{GM/a}$. Using the center-of-mass we can write $a_1/a_2 = v_1/v_2 = m_2/m_1$, $a_1/a = v_1/v = m_2/M$.

III.6.2 $N > 2$

No general solution. Can be chaotic, not purely periodic, not closed. E.g., Jupiter pertburbs the orbit of the Earth, asteroids, comets, slingshots.

How high does N get? Globular cluster: $N \sim 10^6$ stars. Galaxy cluster: $N \sim 10^3$ galaxies. Universe: $\sim 10^{11}$ galaxies. Need big computers to get approximate solutions.

Lecture IV Doppler Shifts and Waves

Kutner 2.2, 5.2

Light is a wave (and a particle). We talk about waves having wavelength λ and frequency ν: λ is a measure of how far before the wave repeats (in meters or such) and frequency is a measure of how often it repeats (in units of Hz which is 1/s). They are related by the speed of the wave, which for light is c:

$$c = \lambda\nu$$

But, the wavelength that a light is emitted at is not the same as it is absorbed at. It can change if there is relative motion toward or away from the light source. We can this change the Doppler shift, since it shifts the wavelength (or frequency). As long as the velocities are $\ll c$, then:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v}{c}$$

So the shift is $\Delta\lambda$ (Δ usually means change). So if something is moving away from us ($v > 0$), then the wavelength gets longer. Since longer wavelengths are often associated with red, we call this a *red shift*. The opposite, when we go toward a lightsource, is a *blue shift*.

We can also use this in frequency, but the sign is opposite (higher frequency means smaller wavelength, so if the wavelength gets smaller ($\Delta\lambda < 0$) the frequency gets higher ($\Delta\nu > 0$):

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu - \nu_0}{\nu_0} = -\frac{v}{c}$$

Lecture V Extra Solar Planets

Kutner 27.5

This is just how to find them — we will return to talking about the planets themselves later.

How do we find planets around other stars?

- Take pictures? Stars are much brighter, so this is very hard (although it has been done recently in some special cases).
- Main way: Newton

Take a planet in orbit with a star. Mostly the planet moves around the star, but since the mass of the star is finite, it moves a bit too around the center of mass: $v_* m_* = v_p m_p$, so $v_* = v_p(m_p/m_*)$. Even though $m_p \ll m_*$ (so $v_* \ll v_p$) it is still measureable, typically with velocities of a few m/s, via Doppler shifts. We have planets that are similar to Jupiter: $m_p \sim M_{\rm Jupiter} \sim 10^{-3} M_\odot$.

Can also find via eclipses (transits). Here we see the dip in light when a planet goes in front of the star. The amount of light that is lost is $\sim (R_p/R_*)^2$, and since $(R_{\rm Jupiter}/R_\odot) \sim 0.1$, the dip is $\sim 1\%$.

And we can (rarely) see the wobble of the star back and forth during the orbit. Here we use $m_* r_* = m_p r_p$, which gives a wobble of $\ll 1''$.

Lecture VI Tides

Kutner 23.5

Gravity to date has been point masses (or perfect spheres). Not points → tidal forces → not spheres. Tides are from differential forces across an object.

Consider two bits of a bigger thing m_1 and m_2 ($m_1 = m_2$) separated by Δr. $F_1 = GMm_1/r^2$. $F_2 = GMm_2/(r+\Delta r)^2 \approx F_1 + -2GMm/r^3 \Delta r$. It's the extra bit that gives rise to tidal forces, and the $1/r^3$ dependence is pretty general. In the Earth-Moon system, the total force is what keeps the orbit steady. But if you subtract off the forces on the center-of-mass you get the tides, which make a bulge that points at the moon.

Another way to think about it: gravity balances centripetal acceleration $GMm/r^2 = v^2/r = \omega^2 r$ to make a stable orbit. But that is only true at the center of mass: too close and gravity is stronger (bits that are too close get even closer). Too far and gravity is too weak, so bits that are too far get farther. This makes bulges.

Tidal period = $2\times$ forcing periods, from two bulges per rotation (toward and away). Height of tides from Sun $\sim$ that from Moon. With rotation of the Earth, get mostly semi-diurnal (≈ 12.4 h) tides from $P_{\rm rot} = 24$ h, which dominate in Atlantic. But some places hay diurnal tides (complicated interactions between water and gravity).

From the orbit of the moon get spring tides (when lunar lines up with solar) and neap tides (when they cancel out).

From the orbit of the earth get semi-annual tides, since the eccentricity of the Earth is not quite 0.

Tides affect: atmosphere, rock, *ocean*.

$$a_{\rm tide} \sim \frac{2GM_{\rm moon}}{r^3} R_\oplus \sim 2g_\oplus \frac{M_{\rm moon}}{M_\oplus} \left(\frac{R_\oplus}{r}\right)^3$$

But the tidal bulge does not fly away due to this extra acceleration. Instead it makes a bit of extra gravity from the extra mass to cancel it out. The bulge has height h, with extra gravity $g' \sim GM_{\rm bulge}/R_\oplus^2$. We have $M_{\rm bulge} \approx hR_\oplus^2 \rho$, which gives $g' \sim GM_\oplus/R_\oplus^2 (h/R_\oplus) \sim g_\oplus (h/R_\oplus)$ (we have used $\rho R_\oplus^3 \sim M_\oplus$). Setting g' equal to $a_{\rm tide}$, we find $h/R_\oplus \sim 2(M_{\rm moon}/M_\oplus)(R_\oplus/r)^3$.

The size of this bulge from the moon on the Earth is about $10^{-7} R_\oplus \sim 60$ cm. And about 25 cm from the Sun. The Earth on the moon is about 2 m. This is of the right *order*, although the details are hard. For instance, in the Bay of Fundy, the tidal forcing period is the same as the time it takes for the water to slosh around. So you get a resonance, and the tides are really high (up to 9 m).

VI.2 Tidal Evolution

Water sloshes, loses energy (heat) — or generates electricity!

Note that the Earth spins faster (24h) than the Moon's orbit (28d). Friction drags the tidal bulge ahead of the moon. This leads to a net torque that slows down rotation. Conserving L, the moon moves away ($L \propto \sqrt{a}$). This also makes orbits more circular, as tides are stronger for $e > 0$, and synchronized (like the moon is now, with rotation period of the moon equal to its orbital period).

We can measure: the moon mooves away at $\approx\ 3\,cm$/yr. And the Earth day slows down at 0.0016 s/century. Note that the length of a day goes up, the length of a month goes up, but the number of days in a month goes down.

For Pluto+Charon, $P_{\rm orb} = P_{\rm Pluto} = P_{\rm Charon} = 6.4$ d, and $e = 0$. So this has already all happened here.

Weird cases exist, like Mars. Its moon Phobos orbits faster than Mars rotates. So tides pull it in! It will hit in about 50 Myr.

VI.3 Roche Limit

When do tides pull things apart? Simple answer:

$$\frac{Gm}{R^2} \sim \frac{2GM_0}{r^3} R$$

where the thing being pulled apart has mass m, size R, and is held together only by gravity. We take $\rho = m/(4\pi/3R^3)$ and $\rho_0 = M_0/(4\pi/3R_0^3)$. Equating the forces we get roughly:

$$r \lesssim 2^{1/3} \left(\frac{\rho_0}{\rho}\right)^{1/3} R_0$$

Could this have given us Saturn's rings? They are inside such a radius (known as the *Roche Limit*, so large moons would have been pulled apart.

VI.4 Tides and Black Holes

Tides are how black holes kill you. It's not that gravity is too strong, it's that the difference in gravity between different parts of you is too strong.

Lecture VII Stars I

Kutner Chapter 9

What is a star?

- Big ball of gas
- Self-gravity → need high pressure inside to support against collapse
- high pressure → high T inside (basic gas physics)
- high T → emit light

A star's life is a long, losing battle with gravity.

Gravity pulls the parts of the stars in. Why do they not fall? Because pressure pushes them out. Which leads to high T, so the star loses energy. This means we need an energy source to keep the star shining. What is that? Gravity? Chemistry? Fission? Fusion?

VII.2 What supports against gravity?

Let us consider:

1. What if there were no support?
2. What provides the support?
3. How much support is necessary?

If the star is spherical, $g_r = GM(r)/r^2$ where $M(r)$ is the amount of mass contained within r. This is independent of $\rho(r)$, and the matter outside r does not matter.

VII.2.1 What if there were no support?

This is relevant for star formation, supernovae. We have things fall inward with radial velocity v_r and acceleration $a_r = -GM(r)/r^2$. The material falls toward the center taking a *free-fall timescale* $t_{\rm ff}$:

$$|v_r| \sim \frac{r_0}{t_{\rm ff}},\ a_r \sim \frac{v_r}{t_{\rm ff}} \sim \frac{r_0}{t_{\rm ff}^2} \sim \frac{GM_0}{r_0^2}$$

From this we can solve $t_{\rm ff} \sim \sqrt{r_0^3/GM_0} \sim \sqrt{1/G\rho_0}$, where $\rho_0 \sim M_0/r_0^3$ is the mean density. We identify this free-fall timescale as the *dynamical time*. If you do out the math in detail (keeping factors of $4\pi/3$ etc.), you find:

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32}}\frac{1}{\sqrt{G\rho_0}}$$

So the time for collapse only depends on the density, not the size.

Object	r	M	ρ	$t_{\rm dyn}$
Earth	6×10^6 m	$10^{-6}\,M_\odot$	5.5 g/cm^3	~ 10 min
Jupiter	7×10^7 m	$10^{-3}\,M_\odot$	1.3 g/cm^3	~ 10 min
Sun	7×10^8 m	$1\,M_\odot$	1.4 g/cm^3	~ 10 min
White Dwarf	7×10^6 m	$1\,M_\odot$	1.4$\times$10^6 g/cm^3	3s
Neutron Star	10^4 m	$1.4\,M_\odot$	7$\times$10^{14} g/cm^3	0.1 ms

VII.2.2 Support comes from gas pressure

Pressure: resists compression. This comes from the kinetic energy of the gas particles. You can think of them each exerting a little force when they bounce off the walls of a box.

Pressure is force/area. In stars, most of the gas is an *ideal gas*, which means that the particles are all independent of each other. In the kinetic theory of ideal gases, the kinetic energy per particle is $3/2k_BT$ (where k_B is Boltzmann's constant). This is an average. It is both an average over time for 1 particle and an average over all particles at one time. From this we can derive the ideal gas law (in a slightly different form to what you might have seen in chemistry):

$$P = nk_BT$$

$n = N/V$ is the number density (units are m^{-3}), the number of particles N per volume V.

Are there non-ideal gases? Yes, we will discuss later. But these are gases where the particles are correlated (the wavefunctions overlap). They can be:

1. Fermi gases, with $P = P(n)$ only (electrons, protons, *degenerate gases*)
2. Bose gases, with $P = P(T)$ only (photons)

But with ideal gases, we have $P = nk_BT$. We can also write the number density n in terms of the mass density ρ (km/m^3). Then we need to figure out how much the average particle weighs: $n = \rho/\mu m_H$. Here m_H is the mass of hydrogen, and μ is the mean molecular weight. If it's only hydrogen, then $\mu = 1$. If it's helium, then $\mu = 4$. What if it is *ionized* hydrogen? Then you have protons and electrons in equal numbers. The protons have mass $\approx m_H$, but the electrons have much less mass. So the average is $m_H/2$, which means $\mu = 0.5$. But writing the ideal gas law this way gives us:

$$P = \frac{\rho k_BT}{\mu m_H}$$

VII.2.3 How much support is needed?

Here we will derive *Hydrostatic equilibium*. This supports fluid against gravity via a pressure *gradient*. We have the gravitational acceleration $mg = (\rho\Delta r\ A)g = P_{\rm bottom}A - P_{\rm top}A$. Cancelling

A, we find $P_{\rm top} = P_{\rm bottom} + \Delta P = P_{\rm bottom} + (\Delta P/\Delta r)\Delta r$, or:

$$\frac{\Delta P}{\Delta r} = -\rho g$$

This is very general, and applies to stars as well as to atmospheres, oceans, etc. What is means is that P increases as you go down/inward.

What can we do with this? In general we need to know $\rho(r)$ and $g(r)$. But we can make some simplifications. We assume that $\rho \approx M/R^3$ and $g = GM(r)/r^2$. Then we get:

$$\frac{\Delta P}{\Delta r} = -\rho(r)\frac{GM(r)}{r^2}.$$

Furthermore, we will take an average over the whole star, the a difference between the inside ($P = P_c$ at $r = 0$) and the outside ($P = 0$ at $r = R$). So we get $\Delta P/\Delta r \approx (0 - P_c)/(R - 0) = -Pc/R$, with P_c the central pressure. Put this in and you get:

$$P_c \approx \frac{GM^2}{R^4}$$

For the Sun this is close: it gives $10^{14}\,\mathrm{N\,m^{-2}}$, which is a bit of an underestimate. For reference, 1 atmosphere is $10^5\,\mathrm{N\,m^{-2}}$. Relations like these are very important for astronomy where the details are hard but we can make general scaling relations between different quantities.

To go further, we can say that the pressure is related to the temperature through the ideal gas law, $P \propto k_B T$. Putting in our approximate density we get $P = (M/R^3)k_B T/\mu m_H$, or $k_B T \approx GM\mu m_H/R$. Again, the details are hard, but the general expression that $T \propto M/R$ is very useful.

Lecture VIII Stellar Energy

Again, a star's life is a long protracted (but losing) battle with gravity. What can balance it? Fundamentally it is pressure, but what keeps the pressure going?

VIII.2 Virial Theorem

Due to Classius (1870). It concerns bound gravitational systems. In essence, the long-term average of kinetic energy is $1/2$ the average of the potential energy. These can each come from different places:

kinetic can be orbital (motion of blobs or stars) or thermal (random motion of particles)

For only orbital energy, that would be something like the Earth-Moon system. Only thermal would be the insides of stars. Or there are situations with a combination like elliptical galaxies and galaxy clusters.

We denote average by $\langle \ldots \rangle$. So the Virial theorem states:

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle$$

This is a bound system, so we have $\langle U \rangle < 0$. We can then look at the total energy $\langle E \rangle = \langle K \rangle + \langle U \rangle$, and if we substitute we find:

$$\langle E \rangle = \frac{1}{2}\langle U \rangle$$

VIII.3 What Powers the Sun and How Long Will It Last?

Kutner 9.1.2, 9.1.3.

We take the Solar luminosity to be 4×10^{26} W, and try to find a way to get that amount of energy out over a long time.

The first estimate was due to Lord Kelvin (1862, in Macmillan's Magazine). This estimate (known now at the Kelvin-Helmholtz time, $t_{\rm KH}$) was shown to be < 100 Myr. But Darwin said (at the time) that fossils were at least 300 Myr old. So something weird was going on. Kelvin's estimate may have been wrong by a bit, but it couldn't be that bad. So there had to be some unknown energy source.

The lifespan of the Sun could be due to:

1. Chemical energy
2. Gravitational energy
3. Thermal energy (could it have just been a lot hotter in the past?)

4. Fission?

The answers for all of these are no. Kelvin's estimate concerned specifically gravitational. Chemical energy isn't enough, since we know about how much chemical energy a given reaction can release for a given amount of stuff. Same with fission.

VIII.3.1 Gravito-Thermal Collapse, or the Kelvin-Helmholtz Timescale

This ascribes the luminosity to the change in total energy: L is change in $E = K + U$.

If you do this you get a timescale of $t_{\mathrm{KH}} \sim 10^7$ yr, which is $\gg t_{\mathrm{ff}}$:

$$t_{\mathrm{KH}} \sim \frac{E}{L}$$

But $E \sim GM_\odot^2/R_\odot \sim 10^{41}\ \mathrm{J} = 10^{48}\ \mathrm{erg}$ (1 J=10^7 erg).

That is because as collapse occurs, $|U|$ increases so K increases too. That heats up the star, which slows down the collapse.

We can use the Virial theorem to get the central temperature T_c of the Sun. We assume that the center (the hottest/densest bit) dominates K:

$$K \sim \frac{3}{2} k_B T_c \frac{M}{\mu m_H}$$

And $K = -U/2$, with $U \sim -GM_\odot^2/R_\odot$. So we find $T_c \sim GM_\odot m_H / k_B R_\odot \sim 10^7$ K. This is pretty good (the real number is about 1.6×10^7 K).

Lecture IX To Make A Star

1. Support against gravity (P from HSE, E from Virial theorem)
2. Source of energy: nuclear

We can exclude all forms of energy besides nuclear fusion from powering the Sun. How does fusion work?

IX.2 Fusion

Kutner 9.3

What this boils down to is $E = mc^2$: if you can get rid of a bit of mass, you liberate a lot of energy.

atomic unit $u = 1.66054 \times 10^{-27}$ kg (mass of ^{12}C/12)

proton $m_p = 1.6726 \times 10^{-27}$ kg = $1.007u = 938.8\,$MeV/c^2

neutron $m_n = 1.6749 \times 10^{-27}$ kg = $1.0087u$

electron $m_e = 9.1 \times 10^{-31}$ kg = $0.0055u$

hydrogen $m_H = 1.0078u = m_p + m_n - \mathrm{electrostatic binding energy}/2$

He nucleus $m_\alpha = 4.002u = 2m_p + 2m_n - \Delta m$, with $\Delta m = 0.03u \sim 0.7\% \times (4m_H) \approx 28\,\mathrm{MeV}/c^2$

So going from 4 protons to 1 He nucleus (α particle) releases 28 MeV. This is the energy released by fusion.

We can think of the binding energy as the energy released when you form something (a nucleus in this case), or as the energy that is required to break something up.

^{1}**H** : $E_b = 0$

^{4}**He** : $E_b = 28\,$MeV = $7.08\,$MeV/nucleon

^{16}**O** : $E_b = 7.97\,$MeV/nucleon

^{56}**Fe** : $E_b = 8.798\,$MeV/nucleon

^{238}**U** : $E_b = 7.3\,$MeV/nucleon

^{56}Fe has the highest binding energy, so it's the most stable. Elements that are lighter or heavier are less stable. This means that reactions would naturally squeeze lighter elements together into Fe (fusion) and break heavier elements apart (fission).

IX.2.1 Basic Nuclear Physics

1. Binding Energy: ${}^A_Z X$, with A the number of nucleons, and Z the number of protons. $E_b = (Zm_p + (A-Z)m_n - m_{\rm nuc})c^2$
2. Strong force: binds nuclei together against Coulomb (electrostatic) repulsion (since protons are positively charged)
3. $A \lesssim 56$: strong force increases faster when A increases than Coulomb forces, so a larger A leads to nuclei that are more bound.
4. $A \gtrsim 56$: the opposite

So fusion builds nuclei up to Fe, while fission breaks them down.

IX.2.1.1 H → Fe

This gives about 9 MeV/nucleon. Going from H to He gets 7 (or about 0.7% of mc^2). Going to O gets about 8 (0.8%). Going to Fe gets about 1% of mc^2 which is the most that fusion can do.

So for each proton you get $\approx 1\% mc^2 \sim 10^{-12}$ J (which means that 1 g of H could supply the annual energy of an american).

Fusion in the Sun: 10^{-12} J $\times M_\odot/m_p \sim 10^{45}$ J $\gg GM_\odot^2/R_\odot$. $t_{\rm nuc} \sim E/L_\odot \sim 10^{11}$ yr, so the Sun could shine for that long.

The actual lifespan is about 10^{10} yr (and it's lived about half of that) for a few reasons:

- $L_\odot$ increases later in life
- Not all H is burned
- It does not get hot enough to burn all the way to Fe

But it is clear that $t_{\rm nuc} \gg t_{\rm KH} \gg t_{\rm dyn}$

Lecture X How to Power the Sun

The basics are $4 \times^1 \mathrm{H} \rightarrow^4 \mathrm{He} + 28\,\mathrm{MeV}$, or releasing $\sim 7\,$MeV/A. But this has some problems.

X.1.2 Problem 1:

coulomb repulsion is strong. In order to have fusion you have to force together multiple hydrogen nuclei. These are protons, and are all positively charged. The strong force can only overcome the repulsion when the protons are *very* close together: $\sim 1\,$fm$= 10^{-15}\,$m (for comparison, an electron orbits at $10^{-11}\,$m).

The *classical* (not quantum) solution to this is that protons get close just because of their motion. They are hot, so they zip around pretty quickly. Sometimes they will approach each other, and this may happen. Can we tell how much?

The Coulomb potential is: $U_C = \frac{1}{4\pi\epsilon_0}\frac{e_1 e_2}{r}$, and energy will be conserved when the protons approach. So if they are travelling fast far away, as they approach the potential barrier they slow down:

$$\frac{1}{2}m_p v_\infty^2 + U_C(\infty) = U_C(1\,\mathrm{fm})$$

taking the limiting case that they have used all of their kinetic energy to get close enough. This gives us a requirement:

$$\frac{1}{2}m_p v_\infty^2 \geq \frac{e^2}{4\pi\epsilon_0(1\,\mathrm{fm})}$$

where we can also relate $\frac{1}{2}m_p v_\infty^2 = \frac{3}{2}k_B T$. So we need

$$T \geq T_{\mathrm{classical}} = \frac{e^2}{6k_B\pi\epsilon_0(1\,\mathrm{fm})} \sim 10^{10}\,\mathrm{K}$$

This is pretty hot, given that we know $T_c \sim 10^7\,$K. So the center of the Sun is not hot enough to sustain nuclear fusion!?

In fact, Arthur Eddington proposed nuclear energy as a power source, but others thought stars were not hot enough. Eddington said: “I am aware that many critics consider the stars are not hot enough. The critics lay themselves open to an obvious retort; we tell them to go and find a hotter place.”

In the end, Eddington was right!

X.1.3 Quantum Mechanics

The problem is that the temperature above was the *classical* result, but quantum mechanics are important here. In particular, we need to consider wave-particle duality and the Heisenberg uncertainty principle: the size of a particle depends on its momentum. Classically you could think of a particle at a certain place with velocity v (and hence momentum mv); but in a quantum sense you need to consider that the particle's position is only known to a de Broglie wavelength $\lambda_B \sim h/p \sim h/mv$, where h is Planck's constant.

When two particles are within λ_B of each other, there is a finite probability that they will "tunnel" to within 1 fm of each other ("tunneling" through the potential barrier):

$$U_c = \frac{e^2}{4\pi\epsilon_0\lambda_B} \le \frac{1}{2}m_p v_\infty^2 = \frac{1}{2}\frac{p^2}{m_p} \sim \frac{1}{2m_p}\left(\frac{h}{\lambda_B}\right)^2$$

Note that we have replaced 1 fm with λ_B. We then get a constraint on λ_B of:

$$\lambda_B < \frac{4\pi\epsilon_0 h^2}{2e^2 m_p} \sim 10^{-13}\,\mathrm{m} = 100\,\mathrm{fm}$$

which is much bigger than the classical result. So we can be $100\times$ further away and still have fusion. This gives us a much gentler requirement for the temperature as well:

$$\frac{3}{2}k_B T = \frac{1}{2}m_p v_\infty^2 = \frac{1}{2m_p}\left(\frac{h}{\lambda_B}\right)^2$$

Which gives:

$$T \ge T_{\mathrm{quantum}} \sim \frac{m_p e^4}{12\pi^2\epsilon_0^2 k_B h^2} \sim 10^7\,\mathrm{K}$$

which is OK!

So fusion is possible because of *quantum tunneling* at 10^7 K for the Sun. But, if $M < 0.08\,M_\odot$, then it cannot even get this hot and fusion is impossible. Such objects are failed stars known as *brown dwarfs*.

Quantum tunneling being possible does not mean that it always happens. The probability is $\sim e^{-2\pi^2 U_c/k_B T}$. So for 10^7 K, the probability is only 10^{-8} for any two protons, and it goes up to 1 for 10^{10} K. But there are enough protons and the fusion rate increases quickly with temperature that you have *ignition* at 10^7 K.

X.1.4 How Often Do Protons Get Close Enough?

At the center of the Sun the density is $\rho \sim 100\,\mathrm{g\,cm^{-3}}$, so the mean separation $l \sim n^{-1/3}$ is $\sim 10^{-11}$ m which is $\gg \lambda_B$. To have fusion with $4 \times {}^1\mathrm{H} \to {}^4\mathrm{He}$ we need 4 protons to get very close together, which is hard.

This reaction is not one that needs 4 protons at once, but it is actually a sequence of 2-body reactions. At "low" temperatures it is the *proton-proton* chain (p-p chain), which at "higher temperatures" there is the *carbon-nitrogen-oxygen* (CNO) chain. This does not burn CNO, but uses them as catalysts.

For later fusion reactions to build heavier elements, the higher charges on the nuclei (more protons) need higher temperatures to get close enough. For instance, to fuse He need 10^8 K. Others need close to 10^9 K. These will happen after the central H has been consumed.

X.1.4.1 p-p chain

The main sequence in the Sun. The net reaction is:

$$4^1\mathrm{H} \rightarrow^4 \mathrm{He} + 2e^+ + 2\nu_e + 2\gamma$$

Which is actually:

$$\begin{aligned}
^1\mathrm{H} +^1 \mathrm{H} &\rightarrow {}^2\mathrm{H} + e^+ + \nu_e \\
^1\mathrm{H} +^1 \mathrm{H} &\rightarrow {}^2\mathrm{H} + e^+ + \nu_e \\
^2\mathrm{H} +^1 \mathrm{H} &\rightarrow {}^3\mathrm{He} + \gamma \\
^2\mathrm{H} +^1 \mathrm{H} &\rightarrow {}^3\mathrm{He} + \gamma \\
^3\mathrm{He} +^3 \mathrm{He} &\rightarrow {}^4\mathrm{He} + 2^1\mathrm{H}
\end{aligned}$$

X.1.4.2 CNO chain

For more massive stars:

$$\begin{aligned}
^{12}\mathrm{C} +^1 \mathrm{H} &\rightarrow {}^{13}\mathrm{N} + \gamma \\
^{13}\mathrm{N} &\rightarrow {}^{13}\mathrm{C} + e^+ + \nu_e \\
^{13}\mathrm{C} +^1 \mathrm{H} &\rightarrow {}^{14}\mathrm{N} + \gamma \\
^{14}\mathrm{N} +^1 \mathrm{H} &\rightarrow {}^{15}\mathrm{O} + \gamma \\
^{15}\mathrm{O} &\rightarrow {}^{15}\mathrm{N} + e^+ + \nu_e \\
^{15}\mathrm{N} +^1 \mathrm{H} &\rightarrow {}^{12}\mathrm{C} +^4 \mathrm{He}
\end{aligned}$$

Notice that the CNO all stay the same throughout: anything that is produced is consumed and vice versa.

Lecture XI The Nucleus

We want to explain the binding energy in $m = Zm_p + (A - Z)m_n - E_b/c^2$, where the nucleus has Z protons and $A - Z$ neutrons, for a total number A nucleons.

XI.2 The Liquid Drop Model

$$E_B \approx a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \delta(A, Z)$$

Let's look at each term:

$a_V A$: this is a volume term, since for constant density nucleons the volume will be $\propto A$. This covers the binding due to eht strong force, which is $\propto A$

$a_S A^{2/3}$: this is a surface term. For a volume $\propto A$, the surface area will be $\propto A^{2/3}$. It works as a correction to the volume term since the nucleons near the outside will have fewer other nucleons to interact with.

$a_C \frac{Z(Z-1)}{A^{1/3}}$: this is the Coulomb term, showing the strength of electrostatic repulsion which is $\propto 1/r \sim 1/A^{1/3}$

$a_A \frac{(A-2Z)^2}{A}$: this is an assymetry term, where nuclei with $A \approx 2Z$ are more bound

$\delta(A, Z)$: this is a pairing term

Overall this makes stable nuclei with $Z \approx A/2$, and says that the most bound nuclei are near $A = 60$. It is obviously a simplification, but it can be improved with the addition of a *shell model* (like for electrons) and using empirical data to set the various constants.

Lecture XII Radiation

Astronomy is based (so far) on observing light from objects. This means we see photons (electromagnetic radiation). Which give us information about temperature, density, chemical composition, etc.

We will discuss:

1. Diffusion and random walks
2. Blackbodies and temperature
3. Photospheres and energy transport

XII.2 Photons

Kutner 2.2, 2.4

Photons are light particles, but they also behave as waves. Each photon has energy $E = h\nu$ (with ν the frequency in Hz) or $E = hc\lambda$ (with λ the wavelength). This comes from the definition $c = \lambda\nu$.

We divide up the electromagnetic spectrum into wavelength regions:

γ-rays : $\lambda < 0.01\,\mathrm{nm}$

X-rays : $1\,\mathrm{nm} \rightarrow 10\,\mathrm{nm}$

ultraviolet (UV) : $10\,\mathrm{nm} \rightarrow 400\,\mathrm{nm}$

optical (visual) : $400\,\mathrm{nm} \rightarrow 700\,\mathrm{nm}$

infrared (IR) : $700\,\mathrm{nm} \rightarrow 1\,\mathrm{mm}$

radio : $> 1\,\mathrm{mm}$

XII.3 Diffusion & Random Walks

Heat (energy) is produced at the centers of stars through nuclear reactions. It escapes ultimately as photons. How long does that take? A naive answer is $\sim R_\odot/c = 2\,\mathrm{s}$.

But that is wrong (although it is true for neutrinos). It actually takes $\sim 10^7$ yrs. Why? Because a star is a very crowded place, and photons (even though they move fast) cannot move very far before they wack into something else and end up going in another direction. They easily bounce (scatter) off of ions, electrons, and atoms, and even other photons.

Each bounce tends to make the photon lose energy, but more photons are then produced, conserving energy. In the center the photons start out as X-ray photons, but by the time they get to the surface of the star they are optical photons. They get there via a *random walk*.

Assume that a photon will move (on average) a distance $l_{\rm mfp}$ before it hits something and changes direction. That distance is the *mean free path*. It travels a distance d after N collisions. We can determine what $d(N)$ is. Assume each one moves $\vec{l_i}$ for $i = 1 \ldots N$, with $|\vec{l_i}| = l_{\rm mfp}$. So the total distance is the vector sum:

$$\vec{d} = \sum^{N} \vec{l_i}$$

We want the magnitude of this, $|\vec{d}| = \sqrt{\vec{d} \cdot \vec{d}}$. But

$$\vec{d} \cdot \vec{d} = \sum_{i}^{N} \vec{l_i} \cdot \vec{l_i} + \sum_{i \neq j} \vec{l_i} \vec{l_j}$$

The second term there will go to 0 on average, since the directions are different. So $|\vec{d}|^2 = N|\vec{l}| = N l_{\rm mfp}$, or $d = \sqrt{N} l_{\rm mfp}$. This is in fact a general result with applicability to a wide range of areas.

From this we can determine how long does it take for a photon to diffuse out of the star. To go a distance d, it takes:

$$\begin{array}{rcll} & & \frac{d}{c} & l_{\rm mfp} > d \\ N\frac{l_{\rm mfp}}{c} & = & \frac{d^2}{l_{\rm mfp} c} & l_{\rm mfp} < d \end{array}$$

This is also often referred to as a “drunkard’s walk”.

XII.4 What Happens When A Photon Hits Something?

- Photon A generates an oscillating electromagnetic field
- Matter (ion, electron, atom) is shaken by that field (absorbing the photon)
- But this shaking is itself a fluctuating field, so it makes a new EM field, releasing photon B

There are 3 basic types of interactions:

1. Scattering: A and B have the same energy (or frequency) but different directions. So the matter gains momentum but no energy
2. Absorption: no photon B is emitted. Matter absorbs energy and generally something happens to it
3. Emission: matter emits B (and it can be with or without A)

We take each obstacle as having cross-sectional area σ (in units of m^2). And we have n of them per m^3 (number density n). So what is $l_{\rm mfp}$?

1. A dimensional analysis: $l_{\rm mfp}$ [m] could be $1/\sigma n$, $\sigma^2 n$, $n^{-1/3}$, ...

2. Physical: shoot a bullet into a box of total area A and depth l_{mfp}. It has N balloons in it each with area σ. The bullet is likely to hit a balloon if $N\sigma = A$. $N = n \times \mathrm{volume} = nl_{\mathrm{mfp}}A$. Equating $Nl_{\mathrm{mfp}}A\sigma = A$ gives $l_{\mathrm{mfp}} = 1/n\sigma$

3. How big are the balloons (what is σ)?

   electrons $10^{-28}\,\mathrm{m}^2$

   H atoms $10^{-20}\,\mathrm{m}^2$

   So for $n \sim 10^{30}\,\mathrm{m}^{-3}$ (which you get for $\rho = 10^3\,\mathrm{kg\,m}^{-3}$ like for water), we find $l_{\mathrm{mfp}} \sim 10^{-11}$ m to 10^{-2} m, both of which are $\ll R_\odot$. Therefore there are many bounces:

   electrons : $l_{\mathrm{mfp}} \sim 10^{-2}$ m, so $t \sim 5000\,\mathrm{yr}$ (10^{22} bounces)

   H atoms : $l_{\mathrm{mfp}} \sim 10^{-11}$ m, so $t \sim 5 \times 10^{12}$ yr (10^{40} bounces)

   The actual answer is about 10^7 yrs, taking into account the changing structure and density of the Sun.